Stochastic Choice and Noisy Beliefs in Games: an Experiment*

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Abstract

We study an equilibrium model in which players make stochastic choices given their beliefs and there is noise in the beliefs themselves. The model primitives are an action-map, which determines a distribution of actions given beliefs, and a belief-map, which determines a distribution of beliefs given opponents’ behavior. These are restricted to satisfy axioms that are stochastic generalizations of “best response” and “correct beliefs”, respectively. In our laboratory experiment, we collect actions data and elicit beliefs for each game within a family of asymmetric 2-player games. These games have systematically varied payoffs, allowing us to “trace out” both the action- and belief-maps. We find that, while both “noise in actions” and “noise in beliefs” are important in explaining observed behaviors, there are systematic violations of the axioms. In particular, although all subjects observe and play the same games, subjects in different roles have qualitatively different belief biases. To explain this, we argue that the player role itself induces a higher degree of strategic sophistication in the player who faces more asymmetric payoffs. This is confirmed by structural estimates.

Keywords: beliefs; quantal response equilibrium; noisy belief equilibrium

JEL Classification: C72, C92, D84

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1 Introduction

Nash equilibrium is the central concept of game theory. It describes a situation of stability in which (i) players best respond to their beliefs over opponents’ behavior and (ii) these beliefs are correct. However, both of these deterministic assumptions are unrealistic in many contexts.

The aim of this paper is to understand the ways in which beliefs and actions deviate from the assumptions of Nash equilibrium. Since the deterministic assumptions of Nash will be trivially rejected, we first characterize as a benchmark model a generalization that allows for both stochastic choice given beliefs and randomness in the beliefs themselves. This model is based on four natural axioms which represent stochastic generalizations of “best response” and “correct beliefs”. Next, we collect experimental data—actions and elicited beliefs—in order to test these axioms. While we find evidence for stochasticity in both actions and beliefs, there are systematic violations of the axioms. We show that these failures are qualitatively consistent with non-linearities in the utility function and an effect of the player role itself on subjects’ strategic sophistication. This is confirmed by estimates of a unified structural model applied to actions and belief statements jointly.

Existing equilibrium models that incorporate stochastic elements have had success in explaining deviations from Nash. Most notably, quantal response equilibrium (QRE) (McKelvey and Palfrey [1995]), which allows for “noise in actions” while maintaining correct beliefs, has become a standard tool for analyzing experimental data. More recently, Friedman [2019] introduced noisy belief equilibrium (NBE), which is shown to explain several of the same phenomena as QRE by injecting “noise in beliefs” while maintaining best response. Both models, however, make some unrealistic predictions that are directly related to the fact of having noise in only one of actions or beliefs. Despite this, the equilibrium effects of allowing both sources of noise have not been examined.

Our first contribution is to introduce a model that allows for noise in both actions and beliefs, which will serve as our benchmark. The model, which we call QNBE, nests QRE and

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1 Even one failure to best respond or any variance in beliefs is inconsistent with the model’s assumptions.

2 NBE shares much of the same structure as random belief equilibrium (Friedman and Mezzetti [2005]), but differs in that the belief distributions are restricted to satisfy behavioral axioms which gives rise to these predictions.

3 For instance, Friedman [2019] shows that QRE cannot be invariant to both scaling and translating payoffs, and, in order to explain observed deviations from Nash equilibrium within individual games (e.g. as documented in McKelvey et al. [2000]), QRE implies an oversensitivity to affine transformations. On the other hand, NBE is invariant to affine transformations but implies that non-rationalizable actions are played with probability zero, which is rejected in many datasets.
NBE. It is defined by an *action-map* which determines the mixed actions taken by players given their beliefs, and a *belief-map* that determines the distribution of players’ beliefs (i.e. a distribution over opponents’ mixed actions) as a function of the opponents’ mixed actions. The action-map is restricted to satisfy the axioms of regular QRE (Goeree et al. [2005]), requiring that, for any given belief, higher payoff actions are played with higher probability (*monotonicity*) and that an all-else-equal increase in the payoff to some action increases the probability that action is played (*responsiveness*). The belief-map is restricted to satisfy the axioms of NBE, requiring that belief distributions are unbiased (*unbiasedness*) and shift (in the sense of stochastic dominance) in the same direction as changes in the opponents’ behavior (*belief-responsiveness*).

As we illustrate through examples, QNBE does impose testable restrictions in standard actions data, but it is fairly permissive in games for which optimal actions depend on beliefs. Hence, using actions data alone, the test of the model would be weak. Moreover, even if we did find a rejection in actions data, this would not pin down which axiom is violated. To resolve this, we elicit beliefs directly, which allows us to identify both the action- and belief-maps without strong auxiliary assumptions. Using this augmented data, we test the axioms. Now, even if the actions data can be rationalized as QNBE outcomes, we may still reject the model and name the offending axiom.

In our second contribution, we run a laboratory experiment in which subjects choose actions and we directly elicit beliefs for a series of games with systematically varied payoffs. This allows us to observe multiple points on (or “trace out”) the empirical action-map and belief-map. Using these maps, we (i) test the axioms, (ii) offer explanations to the extent that the axioms fail, and (iii) quantify the relative importance of action- and belief-noise in explaining features of the data.

Central to our design are the $2 \times 2$ asymmetric matching pennies games whose payoffs are in Table 1. Indexed by different values of player 1’s payoff parameter $X > 0$, these $X$-games have unique mixed strategy Nash equilibria. By varying $X$, QNBE predicts variation in actions and beliefs for both players so that we may observe multiple points on the empirical action- and belief-maps. This is important because some of the axioms cannot be falsified otherwise, and violations of axioms may be local to particular regions of the domain.

In addition to the $X$-games, which are our focus, we also include some dominance solvable games. We use these to derive a subject-level measure of strategic sophistication that helps

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4Similar games were played in the lab for the first time in Ochs [1995].

5We also include a small number of additional games whose data we do not analyze in this paper.
Table 1: Game X. Player 1’s payoff parameter X controls the asymmetry of payoffs. We assume X > 0 which ensures a unique, fully mixed Nash equilibrium.

to rationalize our findings on beliefs.

At the beginning of our experiment, subjects are sorted into player roles (row or column), which they maintain throughout. Subjects state beliefs and take actions for games that appear in random order. These include the X-games for six different values of X. At no point do subjects receive feedback, and each game appears several times so that we may observe multiple elicitations per subject.

In testing the axioms, we find that comparative statics (responsiveness and belief-responsiveness) hold, but restrictions on levels (monotonicity and unbiasedness) do not. For the axioms that are rejected, our findings differ across player roles.

Consistent with responsiveness, we find that an increase in the expected payoff to an action (through variations in beliefs for a given game) increases the probability the action is played. This is true for both players, all games, and for all regions of expected payoffs.

In testing monotonicity, we find systematic failures for player 1 only: for each game, there is an interval of beliefs to which subjects fail to best respond more often than not. These intervals involve beliefs for which the action that has a higher expected payoff is also more likely to result in a zero payoff.

Consistent with belief-responsiveness, we find that player i’s belief distributions tend to be ordered by stochastic dominance across games in the same direction as differences in player j’s action frequencies. Beliefs tend to overreact in the sense that small differences in action frequencies are associated with large differences in average beliefs, but this is entirely consistent with the axiom.

In testing unbiasedness, we find that player 1 is marginally biased, tending to form slightly conservative beliefs that are closer to the uniform distribution than player 2’s actual frequency of play. Player 2, on the other hand, forms very biased, extreme beliefs: whereas player 1’s behavior is relatively close to uniform across all X-games, player 2 tends to think that player 1 will overwhelmingly choose U when X is large and similarly choose D when
$X$ is small. Whereas conservative beliefs have been found in games played without feedback (e.g. Huck and Weizsacker [2002]), we believe this asymmetric pattern of bias is novel.

This gives two puzzles with respect to the benchmark QNBE model. These are a failure of *monotonicity* for player 1-subjects who fail to best respond more often than not given some intervals of beliefs and a failure of *unbiasedness*, with the nature of bias depending on player role. We provide explanations and a fitted model that can capture these features of the data.

To explain the failure of *monotonicity* for player 1, we show that, given stated beliefs, concavity in the utility function over payoffs qualitatively predicts precisely the violations we observe (payoffs are in probability points of earning a prize, so this is distinct from risk aversion). This is backed by structural estimates, which suggest that most subjects individually have concavity and that a reasonable calibration can accommodate most of the violations.\(^6\)

To explain the failure of *unbiasedness*, our first clue is that the belief-bias is qualitatively different for the two players. This leads us to conjecture that one player, by merit of her role in the game, is induced to think about her opponent more deeply or with greater “strategic sophistication”. This could generate the bias as player 2 believes that player 1 tends to take the low-level action (the best response to random behavior: $U$ when $X$ is large, $D$ when $X$ is small) whereas player 1 anticipates this and acts accordingly.

This sophistication hypothesis cannot be tested within the $X$-games directly, but can be studied with the help of the dominance solvable games. All subjects face the same action and belief choices in these games, so we can use the belief statements to derive a subject-level measure of strategic sophistication that is collected identically for all subjects. We formally justify this measure through a level $k$-type framework.\(^7\) Using this measure, we find that player 1-subjects have much higher levels of sophistication than player 2-subjects. All subjects see exactly the same games throughout the experiment, were randomly assigned to their roles, and played a number of $X$-games before playing a dominance solvable game. Hence, we conclude that experience in the player 1-role of the $X$-games causally induces greater sophistication and this somehow spills over to the dominance solvable games.

Based on these sophistication results, we consider generalized level $k$-type models to rationalize the beliefs data. The simplest model that explains the large majority of individual

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\(^6\) Previous studies eliciting beliefs in games (Costa-Gomes and Weizsacker [2008] and Rey-Biel [2009]) find little evidence of risk aversion though there are exceptions (Ivanov [2011]).

\(^7\) The level $k$ literature was started by Nagel [1995] and Stahl and Wilson [1995], and is reviewed in Crawford et al. [2013].
subjects’ belief patterns is a parametric, subjective cognitive hierarchy model (Camerer [2003]) that embeds payoff sensitivity (as in QRE) into players’ models of others. We fit the model to individual subjects’ belief data from the entire set of X-games. We interpret one of the fitted parameters as strategic sophistication, and find that this model also captures the sophistication gap between the players. The model predicts that this inferred sophistication should be correlated at the subject level with the measure derived from the dominance solvable games. We find very strong correlations, which we take as further evidence that player 1-subjects’ beliefs in the X-games indicate higher levels of sophistication.

Next, we conduct a counterfactual exercise to determine the relative importance of action-noise and belief-noise for explaining the data. Specifically, we consider (i) the action frequencies that we would have observed if subjects best responded to all of their stated beliefs and (ii) the action frequencies we would have observed if subjects had beliefs equal to their opponents’ empirical action frequencies (and their actions were determined by a best-fit random utility model). These correspond to “turning off” action-noise and belief-noise, respectively. Both counterfactuals deviate considerably from the empirical action frequencies, indicating that both sources of noise are important. Comparing the performance of the counterfactuals, we find that the latter is more accurate for player 1 (i.e. action-noise is more important) and the former is more accurate for player 2 (i.e. belief-noise is more important). Hence, ignoring any one source of noise may lead to misspecification, and which source of noise is more important depends on the context.

Our analysis throughout the paper implicitly assumes that stated beliefs equal the underlying “true” beliefs that subjects hold in their minds and guide their actions. This is, of course, a hypothesis that cannot be directly tested (see Rutstrom and Wilcox [2009] for a discussion). Since we take “noise in beliefs” seriously, we consider the possibility that stated beliefs are simply noisy signals of true beliefs. Assuming this were the case, can we reject the axioms with respect to true beliefs? Could we say that true beliefs are noisy at all? We show that, under mild assumptions, the answer to both questions is yes.

This paper contributes to a large literature on behavioral game theory (Camerer et al. [2004a]) that focuses on bounded rationality. More narrowly, we contribute to the theory and empirical study of equilibrium models that inject stochasticity into actions and beliefs (especially McKelvey and Palfrey [1995], Chen et al. [1997], Friedman and Mezzetti [2005], and Friedman [2019]). Our central tool is belief elicitation, so we engage with the growing methodological literature on belief elicitation (see Schotter and Trevino [2014] and Schlag et al. [2015] for review articles) and benchmark our findings against those from well-known
studies that elicited beliefs (e.g. Nyarko and Schotter [2002], Costa-Gomes and Weizsacker [2008] and Rey-Biel [2009]). Our key innovation is to collect multiple elicitations per subject without feedback for each game within a family of closely related games. This allows us to study noise in beliefs and examine how beliefs vary across games. We refer to Section 9 for a detailed discussion of the literature.

The paper is organized as follows. Section 2 presents the theory, Section 3 gives the experimental design, and Section 4 provides an overview of the data. Section 5 presents the results from testing the axioms, Section 6 offers explanations for the axioms’ failure, and Section 7 studies the relative importance of action- and belief-noise. Section 8 discusses issues of belief elicitation and the interpretation of stated beliefs—and how these may affect the interpretation of our results. Section 9 discusses the relationship of this paper to the existing literature, and Section 10 concludes.

2 Theory

The deterministic assumptions of “best response” and “correct beliefs” implicit in Nash equilibrium will be trivially rejected, so we introduce a new benchmark model that replaces these deterministic assumptions with stochastic generalizations.

The model we study is a hybrid, defined by an action-map satisfying the axioms of regular QRE (Goeree et al. [2005]) and a belief-map satisfying the axioms of NBE (Friedman [2019]). Anticipating the experiment, we present the case of $2 \times 2$ games in which there are two players with two actions each, but as QRE and NBE are defined very generally, the model generalizes to all finite, normal form games.

A game is defined by $\Gamma^{2 \times 2} = \{N, A, u\}$ where $N = \{1, 2\}$ is the set of players, $A = A_1 \times A_2 = \{U, D\} \times \{L, R\}$ is the action space, and $u = (u_1, u_2)$ is a vector of payoff functions with $u_i : A \to \mathbb{R}$. In other words, this is any game in which player 1 can move up (U) or down (D) and player 2 can move left (L) or right (R).

We use $i$ to refer to a player and $j$ for her opponent. We reserve $k$ and $l$ for action indices. Since each player has only two actions, we write player $i$’s mixed action as $\sigma_i \in [0, 1]$. In an abuse of notation, we use $\sigma_1 = \sigma_U$ and $\sigma_2 = \sigma_L$ to indicate the probabilities with which player 1 takes $U$ and player 2 takes $L$, respectively.
2.1 Action-map

Let $\sigma'_j \in [0, 1]$ be an arbitrary belief that player $i$ holds over player $j$’s action. Given this belief, player $i$’s vector of expected payoffs is $\bar{u}_i(\sigma'_j) = (\bar{u}_{i1}(\sigma'_j), \bar{u}_{i2}(\sigma'_j)) \in \mathbb{R}^2$, where $\bar{u}_{ik}(\sigma'_j)$ is the expected payoff to action $k$. We use $v_i = (v_{i1}, v_{i2}) \in \mathbb{R}^2$ as shorthand for an arbitrary such vector. That is, $v_i$ is understood to satisfy $v_i = \bar{u}_i(\sigma'_j)$ for some $\sigma'_j$.

As in QRE, the action-map is induced by a quantal response function $Q_i : \mathbb{R}^2 \to [0, 1]$. This maps any vector of expected payoffs (given beliefs) to a mixed action, and it is assumed to satisfy the following regularity axioms (Goeree et al. [2005]):

(A1) **Interiority**: $Q_{ik}(v_i) \in (0, 1)$ for all $k \in 1, 2$ and for all $v_i \in \mathbb{R}^2$.

(A2) **Continuity**: $Q_{ik}(v_i)$ is a continuous and differentiable function for all $v_i \in \mathbb{R}^2$.

(A3) **Responsiveness**: $\frac{\partial Q_{ik}(v_i)}{\partial v_{il}} > 0$ for all $k \in 1, 2$ and $v_i \in \mathbb{R}^j(i)$.

(A4) **Monotonicity**: $v_{ik} > v_{il} \implies Q_{ik}(v_i) > Q_{il}(v_i)$ and $v_{ik} = v_{il} \implies Q_{ik}(v_i) = \frac{1}{2}$.

(A1) and (A2) are non-falsifiable technical axioms. Taken together, (A3) and (A4) are a stochastic generalization of “best response”, requiring than an all-else-equal increase in the payoff to an action increases the probability it is played and that, given any belief, the best response is taken more often than not.\(^8\)

2.2 Belief-map

Player $i$’s belief about $j$’s mixed action is drawn from a distribution that depends on $j$’s mixed action. In other words, player $i$’s beliefs are a random variable $\sigma_j^*(\sigma_j)$ whose distribution depends on $\sigma_j$ and is supported on $[0, 1]$. This family of random variables, or belief-map, is described by a family of CDFs: for any potential belief $\bar{\sigma}_j \in [0, 1]$, $F_i(\bar{\sigma}_j|\sigma_j)$ is the probability of realizing a belief less than or equal to $\bar{\sigma}_j$ given that player $j$ is playing $\sigma_j$. Following Friedman [2019], the belief-map is assumed to satisfy the following axioms:

(B1) **Interior full support**: For any $\sigma_j \in (0, 1)$, $F_i(\bar{\sigma}_j|\sigma_j)$ is strictly increasing and continuous in $\bar{\sigma}_j \in [0, 1]$.

(B2) **Continuity**: For any $\bar{\sigma}_j \in (0, 1)$, $F_i(\bar{\sigma}_j|\sigma_j)$ is continuous in $\sigma_j \in [0, 1]$.

(B3) **Belief-responsiveness**: For all $\sigma_j < \sigma'_j \in [0, 1]$, $F_i(\bar{\sigma}_j|\sigma'_j) < F_i(\bar{\sigma}_j|\sigma_j)$ for $\bar{\sigma}_j \in (0, 1)$.

\(^8\)Requiring that $v_{ik} = v_{il} \implies Q_{ik}(v_i) = \frac{1}{2}$ in (A4) is unnecessary since it is implied by $v_{ik} > v_{il} \implies Q_{ik}(v_i) > Q_{il}(v_i)$ and (A1). We added this condition to (A4) in order to have a clean division between technical and behavioral axioms.
(B4) **Unbiasedness:** $F_i(\sigma_j | \sigma_j) = \frac{1}{2}$ for $\sigma_j \in (0, 1)$. $\sigma^*_j(0) = 0$ and $\sigma^*_j(1) = 1$ with prob. 1.

(B1) and (B2) are non-falsifiable technical axioms. (B1) requires that belief distributions have full support and no atoms when the opponent’s action is interior, and (B2) requires that the belief distributions vary continuously in the opponent’s behavior except possibly as the opponent plays a pure action with a probability that approaches one. Taken together, (B3) and (B4) are a stochastic generalization of “correct beliefs”. (B3) requires that, when the opponent’s action increases, beliefs shift up in a strict sense of stochastic dominance.\(^9\) (B4) imposes that belief distributions are correct on median. Both median- and mean-unbiasedness can be microfounded via a model of sampling (Friedman [2019]). The technical axioms allow for either or both types of unbiasedness. We use median-unbiasedness to derive theoretical results because it turns out to be much simpler in our setting, but we test for both types of unbiasedness in our data.

2.3 Equilibrium

In equilibrium, player $i$ quantal responds to belief realizations where the beliefs are drawn from a distribution that depends on $j$’s mixed action—and $j$’s mixed action is her expected quantal response similarly induced by quantal responding to belief realizations.

Given player $j$’s mixed action $\sigma_j \in [0, 1]$, player $i$’s beliefs are drawn according to $F_i(\cdot | \sigma_j)$. For each belief realization $\sigma'_j \in [0, 1]$, player $i$’s mixed action is given by quantal response to expected payoffs $Q_i(\bar{u}_i(\sigma'_j)) \in [0, 1]$. Player $i$’s expected quantal response as a function of $\sigma_j$, which we call the reaction function, simply integrates over belief realizations: $\Psi_i(\sigma_j; Q_i, \sigma^*_j) \equiv \int_{[0, 1]} Q_i(\bar{u}_i(\sigma'_j))dF_i(\sigma'_j | \sigma_j) \in [0, 1]$. Since $Q_i : \mathbb{R}^2 \to [0, 1]$ is single-valued, $\Psi_i$ is also single-valued, i.e. a function as opposed to a correspondence.

A given profile of quantal response functions $Q = (Q_1, Q_2)$ and belief-maps $\sigma^* = (\sigma^*_1, \sigma^*_2)$ induce the reaction function $\Psi = (\Psi_1, \Psi_2) : [0, 1]^2 \to [0, 1]^2$. Equilibrium is defined as a mixed action profile that is a fixed point along with the supporting belief distributions.

**Definition 1.** Fix $\{\Gamma^{2 \times 2}, Q, \sigma^*\}$. A quantal response-noisy belief equilibrium (QNE) is a pair $\{\sigma, \sigma^*(\sigma)\}$ where $\sigma = \Psi(\sigma; Q, \sigma^*)$ is a mixed action profile and $\sigma^*(\sigma)$ is the supporting profile of belief distributions.

The mapping $\Psi$ is continuous, so existence follows from Brouwer’s fixed point theorem.

\(^9\)This is stronger than standard stochastic dominance, which helps with comparative statics and in establishing uniqueness of equilibria, but the distributions can still be arbitrarily close, so it is only slightly stronger.
**Proposition 1.** Fix $\{\Gamma^{2\times2}, Q, \sigma^*\}$. A QNBE exists.

**Proof.** See Appendix 11.2. □

## 2.4 QRE and NBE

QRE is defined as in QNBE except that beliefs are correct with probability one.

**Definition 2.** Fix $\{\Gamma^{2\times2}, Q\}$. A quantal response equilibrium (QRE) is any mixed action profile $\sigma$ such that $\sigma = Q(\bar{u}(\sigma))$.

Similarly, NBE is defined as in QNBE except that players best respond to all belief realizations.

**Definition 3.** Fix $\{\Gamma^{2\times2}, \sigma^*\}$. A noisy belief equilibrium (NBE) is a pair $\{\sigma, \sigma^*(\sigma)\}$ where $\sigma \in \psi(\sigma; \sigma^*)$ and $\psi_i(\sigma_j; \sigma^*_j) \equiv \int_{[0,1]} BR_i(\bar{u}_i(\sigma'_j))dF_i(\sigma'_j|\sigma_j)$ defines the expected best response correspondence.\(^{10}\)

In other words, the QRE belief-map is the identity map and the NBE action-map is the best response correspondence. For almost every game, the sets of attainable QRE and NBE mixed action profiles—that can be supported for some primitives—are nested in the set of attainable QNBE mixed action profiles.\(^{11}\) For the games we analyze in this paper, we show this directly.

## 2.5 $X$-games

We specialize theory for the family of $X$-games whose payoffs are in Table 1. This serves to illustrate the QNBE model and provides our justification for using the $X$-games in the experiment.\(^{12}\)

\(^{10}\)BR\(_i\) is the standard best response correspondence: $BR_i(v_i) = 1$ if $v_{i1} > v_{i2}$, $BR_i(v_i) = 0$ if $v_{i1} < v_{i2}$ and $BR_i(v_i) = [0,1]$ if $v_{i1} = v_{i2}$. $\psi_i$ is the *expected* best response correspondence, where the expectation is over belief realizations whose distribution depends on the opponent’s behavior. Friedman [2019] shows that $\psi_i$ single-valued in generic games since the probability of indifference is zero by (B1).

\(^{11}\)A *non-generic counterexample*. If players are indifferent between all of their actions, independent of their opponents’ behavior, then any distribution of actions is an NBE for any belief-map by (B4). On the other hand, QNBE and QRE predict that all players uniformly mix, and this is true for all primitives by (A4).

\(^{12}\)The results characterizing behavior within a game generalize to all $2 \times 2$ games with unique, mixed strategy Nash equilibria. Comparative static results generalize to any such game with respect to changes in any one payoff parameter.
The X-games have unique, mixed strategy Nash equilibria (NE). As is well-known, NE predicts each player must mix to make the other player indifferent, and so \( \sigma_{L}^{NE,X} = \frac{20}{20+X} \) and \( \sigma_{U}^{NE,X} = 0.5 \) (i.e. constant for all \( X \)). Since we are only working within the X-game family and \( \sigma_{L}^{NE,X} \) is a strictly decreasing function of \( X \), we think of \( \sigma_{L}^{NE,X} \) as a parameter of the game, and we freely go between \( X \) and \( \sigma_{L}^{NE,X} \) as convenient.

First, we establish that, for any fixed primitives, the QNBE is unique.

**Proposition 2.** Fix \( \{X, Q, \sigma^{*}\} \). There is a unique QNBE.

**Proof.** See Appendix 11.2.

There is a unique QNBE for any fixed primitives, but since the primitives are only restricted to satisfy axioms, we characterize the set of equilibria that can be attained for some primitives. The next result characterizes the reaction functions consistent with the axioms and thus the set of mixed action profiles that can be supported as QNBE outcomes. The proof is by construction, and hence implicitly gives the equilibrium belief distributions as well, though we abstract from that here.

**Proposition 3.** Fix \( X \). (i) Any reaction function \( \Psi_{U} : [0, 1] \rightarrow [0, 1] \) that is continuous, strictly increasing, and satisfying the restrictions of (1) can be induced for some primitives \( \{Q_{U}, \sigma_{L}^{*}\} \). (ii) Any reaction function \( \Psi_{L} : [0, 1] \rightarrow [0, 1] \) that is continuous, strictly decreasing, and satisfying the restrictions of (2) can be induced for some primitives \( \{Q_{L}, \sigma_{U}^{*}\} \). (iii) Any \( \sigma = (\sigma_{U}, \sigma_{L}) \) satisfying \( \sigma_{U} \in \Phi_{U}^{X}(\sigma_{L}) \) and \( \sigma_{L} \in \Phi_{L}^{X}(\sigma_{U}) \) can be supported as QNBE outcomes for some primitives \( \{Q, \sigma^{*}\} \).

\[
\Phi_{U}^{X}(\sigma_{L}) = \begin{cases} 
(0, 3/4) & \sigma_{L} < \sigma_{L}^{NE,X} \\
(1/4, 3/4) & \sigma_{L} = \sigma_{L}^{NE,X} \\
(1/4, 1) & \sigma_{L} > \sigma_{L}^{NE,X}
\end{cases} \quad (1)
\]

\[
\Phi_{L}^{X}(\sigma_{U}) = \begin{cases} 
(1/4, 1) & \sigma_{U} < \frac{1}{2} \\
(1/4, 3/4) & \sigma_{U} = \frac{1}{2} \\
(0, 3/4) & \sigma_{U} > \frac{1}{2}
\end{cases} \quad (2)
\]

**Proof.** See Appendix 11.2.

Figure 1 illustrates the proposition for \( X = 80 \), in which case \( \sigma_{L}^{NE,X} = 1/5 \). Here, we plot equilibrium mixed actions in the unit square of \( \sigma_{L} - \sigma_{U} \) space. The first panel plots
\( \Phi^X_U(\sigma_L) \) (1) and the second panel plots \( \Phi^X_L(\sigma_U) \) (2). Where these two regions intersect (third panel) is the set of QNBE mixed action profiles that can be attained for some \( \{Q, \sigma^*\} \) (part \((iii)\) of the proposition).

As shown in Figure 1, the set of attainable QNBE mixed action profiles can be rather large. For \( X = 80 \), the Lebesgue measure is 51.25%, meaning just over half of all possible mixed action profiles can be supported as QNBE outcomes. However, QNBE makes predictions over actions and beliefs, so even if the actions data falls in this region, the axioms—and thus the model—may be falsified.

Friedman [2019] showed that for any \( 2 \times 2 \) game with a unique, fully mixed NE—and hence for any \( X \)-game also—the sets of attainable QRE and NBE mixed action profiles coincide. In Figure 1, we plot this set as a cross-hatched rectangle, which has a measure of 15% (see Goeree et al. [2005] and Friedman [2019] for the derivation of such sets in similar games). Hence, allowing for just one of action-noise or belief-noise leads to the same measure of outcomes, but allowing for both increases the set of outcomes more than 3-fold.

Our next result is a comparative static.

**Proposition 4.** Fix \( \{Q, \sigma^*\} \). \( \sigma^{QNBE}_U \) is strictly decreasing in \( \sigma^{NE,X}_L \) and \( \sigma^{QNBE}_L \) is strictly increasing in \( \sigma^{NE,X}_L \).

**Proof.** See Appendix 11.2. \( \square \)

The proposition says that, under QNBE, varying \( X \) will cause systematic variation in mixed actions, and thus belief distributions also, for both players. This comparative static holds for QRE and NBE also (Goeree et al. [2005] and Friedman [2019]) and is essential in order to “trace out” the empirical action- and belief-maps.

We extend our results to a characterization of QNBE for any finite number of games. To this end, let \( \{\hat{\sigma}^X_U, \hat{\sigma}^X_L\} \) be a dataset of mixed actions from an arbitrary (finite) set of \( X \)-games. It is immediate that in order to support the dataset as QNBE outcomes for some primitives (held fixed across games), it is necessary for the restrictions of Proposition 3 to hold for each \( X \) and the comparative static of Proposition 4 to hold across any pair of \( X \)s. As it turns out, this is also sufficient.

**Proposition 5.** Let \( \{\hat{\sigma}^X_U, \hat{\sigma}^X_L\} \) be a dataset of mixed actions for any finite number of \( X \)-games. The data can be supported as QNBE outcomes for some primitives \( \{\sigma^*, Q\} \) (held fixed across games) if and only if

(i) \( \hat{\sigma}^X_U \in \Phi^X_U(\hat{\sigma}^X_L) \) for all \( X \), where \( \Phi^X_U \) is defined as in (1),

(ii) \( \hat{\sigma}^X_L \in \Phi^X_L(\hat{\sigma}^X_U) \) for all \( X \), where \( \Phi^X_L \) is defined as in (2),
Figure 1: *QNBE in game X = 80*. The first panel gives the region in which player 1’s QNBE reaction must lie, with an example drawn in blue. The second panel gives the region in which player 2’s QNBE reaction must lie, with an example drawn in red. The third panel plots the intersection of the two regions which gives the set of QNBE mixed action profiles that can be attained for some primitives. The black diamond is the Nash equilibrium, the cross-hatched rectangle gives the sets of attainable QRE and NBE, which coincide, and the green dot is an example QNBE mixed action profile.
Figure 2: QRE and NBE in the X-games as a function of $\sigma^{NE}_L$. This figure plots a hypothetical dataset $\{\hat{\sigma}^X_U, \hat{\sigma}^X_L\}_X$ of mixed actions for the set of X-games with $X \in \{80, 40, 10, 5, 2, 1\}$. The left panel plots $\hat{\sigma}^X_U$ and the right panel plots $\hat{\sigma}^X_L$ (green dots), both as functions of $\sigma^{NE}_L$. The data can be supported as QRE or NBE outcomes for some primitives (held fixed across games) if and only if the data is in the gray region, decreasing in the left panel, and increasing in the right panel.

(iii) $\hat{\sigma}^X_U$ is strictly decreasing in $\sigma^{NE,X}_L$, and
(iv) $\hat{\sigma}^X_L$ is strictly increasing in $\sigma^{NE,X}_L$.

Proof. See Appendix 11.2.

Since we are concerned with tracking patterns of behavior across games, we would like a plot to help visualize both the data and model predictions from the entire set of X-games. However, since the set of attainable QNBE mixed action profiles is not rectangular (see Figure 1), it is too cumbersome to plot the QNBE predictions as a function of one variable. For this reason, we provide an analogue of Proposition 5 for QRE and NBE, which are rectangular. The characterizations for both models coincide.

Proposition 6. Let $\{\hat{\sigma}^X_U, \hat{\sigma}^X_L\}_X$ be a dataset of mixed actions for any finite number of X-games. The data can be supported as QRE or NBE outcomes for some primitives (held fixed across games) if and only if

(i) $\hat{\sigma}^X_U \in (1/2, 1)$ for $\sigma^{NE,X}_L < 1/2$; $\hat{\sigma}^X_U \in (0, 1/2)$ for $\sigma^{NE,X}_L > 1/2$,
(ii) $\hat{\sigma}^X_L \in (\sigma^{NE,X}_L, 1/2)$ for $\sigma^{NE,X}_L < 1/2$; $\hat{\sigma}^X_L \in (1/2, \sigma^{NE,X}_L)$ for $\sigma^{NE,X}_L > 1/2$,
(iii) $\hat{\sigma}^X_U$ is strictly decreasing in $\sigma^{NE,X}_L$, and
(iv) $\hat{\sigma}^X_L$ is strictly increasing in $\sigma^{NE,X}_L$. 

13
Proof. See Appendix 11.2.

Figure 2 plots the sets of attainable QRE and NBE as functions of \( \sigma^{NE}_L \), which are given in the proposition. The vertical dotted lines correspond to specific values of \( X \) (marked at the top). We also plot a hypothetical dataset \( \{\hat{\sigma}^X_U, \hat{\sigma}^X_L\}_X \) as green dots: the left panel plots \( \hat{\sigma}^X_U \) and the right panel plots \( \hat{\sigma}^X_L \)—both as functions of \( \sigma^{NE}_L \). The proposition says that a dataset can be supported as QRE or NBE outcomes if and only if it looks qualitatively like the green dots in the figure: in the gray regions, decreasing in the left panel, and increasing in the right.

3 Experimental Design

Recall that the goal of our experiment is to make observable the empirical action- and belief-maps, which we pursue through collecting actions and beliefs data for a family of games. An important consideration is to be able to interpret within-subject variations in actions and beliefs as the result of idiosyncratic “noise” as opposed to other predictable variations.

3.1 Overall structure

The experiment consisted of two treatments, which we label [A, BA] and [A, A]. Our sessions were run in the Columbia Experimental Laboratory in the Social Sciences (CELSS). Subjects were mainly undergraduate students at Columbia and Barnard Colleges, all of whom were recruited via the Online Recruitment System for Economics Experiments (ORSEE) (Greiner [2015]).

The main treatment is [A, BA], which we describe here. The treatment [A, A] is similar, but does not involve belief elicitation. It was included to test whether belief elicitation itself has an effect on behavior, and we defer its discussion to Section 8.

The experiment involved 2 × 2 matrix games, and at the beginning of the experiment, subjects were divided into two equal-sized subpopulations of row and column players, which we refer to as players 1 and 2, respectively. The [A, BA] treatment consisted of two stages. Each round of the first stage involved taking actions, and each round of the second stage involved stating a belief and taking an action. The name of the treatment reflects this (“A” for “action”, “BA” for “belief-action”).

In each of the 20 rounds of the first stage, subjects were anonymously and randomly paired and took actions simultaneously. In each of the 40 rounds of the second stage, subjects were
presented with a payoff matrix that appeared in the first stage. Subjects then stated their beliefs over opponents’ expected action choices before taking actions. These beliefs were elicited over actions taken by subjects in the first stage, and these actions were similarly paired against randomly selected actions (from the relevant games) taken in the first stage. In this way, subjects in the second stage were both forming beliefs about and playing against subjects from the first stage whose actions had already been recorded. Subjects in the second stage were not paired since they were playing against subjects from the first stage. For this reason, subjects in the second stage were not required to wait for all subjects to finish a round before moving on to the next, though in both stages subjects were required to wait for 10 seconds before submitting their answers. Screenshots of the experimental interface are given in Appendix 11.5.

Before the start of the first stage, instructions (see Appendix 11.1) were read aloud accompanied by slides. These instructions described the strategic interaction and taught subjects how to understand $2 \times 2$ payoff matrices. Subjects then answered 4 questions to demonstrate understanding of how to map players’ actions in a game to payoff outcomes. All subjects were required to answer these correctly. Subjects then played 4 unpaid practice rounds before proceeding to the paid rounds. After the first stage, additional instructions for the second stage were given. Only at that point were subjects introduced to the notion of a belief and the elicitation mechanism described. Subjects then played 3 unpaid practice rounds before proceeding to the paid rounds of the second stage.

We are interested in observing the stochasticity inherent in beliefs, not changes in beliefs that are due to new information. For this reason, at no point during the experiment (including the unpaid practice rounds) were subjects provided any feedback. In particular, no feedback was provided about other subjects’ actions, the outcomes of games, or the accuracy of belief statements. Only at the end of the experiment did subjects learn about the outcomes of the games and belief elicitations that were selected for payment. This also simplifies the analysis because subjects could not condition on the history of play.

We also wish to avoid other non-inherent sources of stochasticity in beliefs. Since we elicited beliefs about the first-stage actions which had already been recorded, multiple elicitation for a given game all refer to the same event. Hence, variation in an individual subject’s beliefs for a given game cannot be due to a higher-ordered belief that other subjects were learning. To avoid stochasticity in stated beliefs due to mechanical error, belief

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13 After entering a belief for the first time in a round, subjects could freely modify both their actions and beliefs in any order before submitting. In any case, we see very few revisions of stated beliefs.
statements had to be entered as whole numbers into a box rather than via a slider.

Each game was played multiple times. This was necessary because we wish to analyze stochasticity and patterns in individual subjects’ belief data. However, we took several measures to approximate a situation in which each game was seen as if for the first time. First, there was no feedback, as described. Second, there was a large “cross section”, i.e. more distinct games than the number of times each game was played. Third, the games appeared in a random order which is described in Section 3.2.

In addition to a $10 show-up fee, subjects were paid according to one randomly selected round (based on actions) from the first stage and four randomly selected rounds from the second stage—two rounds based on actions and two rounds based on beliefs (see Section 3.3 for details on belief payments). Since there were twice as many rounds in the second stage as in the first stage, this equated the incentives for taking actions across the stages. Each unit of payoff in the matrix corresponded to a probability point of earning $10 (e.g. 20 is a lottery that pays $10 with probability 20% and $0 otherwise). This was to mitigate the effects of risk aversion as expected utility is linear in probability points. This is important for our purposes since several of our tests require that utilities are identified.

To allay any hedging concerns, all five payments were based on different matrices and this was emphasized to subjects.

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Player 1-subjects</th>
<th>Player 2-subjects</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>[A,BA]</td>
<td>54</td>
<td>56</td>
<td>110</td>
</tr>
<tr>
<td>[A,A]</td>
<td>27</td>
<td>27</td>
<td>54</td>
</tr>
<tr>
<td>Total</td>
<td>81</td>
<td>83</td>
<td>164</td>
</tr>
</tbody>
</table>

Table 2: Overview of experiment.

Table 2 summarizes the number of subjects who participated in the experiment by treatment and player role.15 On average, the experiment took about 1 hour and 15 minutes, and the average subject payment was $19.5.

14Evidence suggests that this significantly, but not completely, linearizes payoffs in the sense that people still behave as if they have a utility function over probability points with some curvature. See for example, Harrison et al. [2012].

15There are two fewer player 1-subjects than player 2-subjects in [A,BA]. This is because two subjects (in separate sessions) had to leave early. They left after the first stage, and since the whole experiment was anonymous and without feedback and the second stage was played asynchronously, this had no effect on the rest of the subjects. These two subjects’ data was dropped prior to analysis.
3.2 The games

As discussed in Section 2.5, the X-games take center-stage since they are predicted to give rise to systematic variation in actions and beliefs. Henceforth, we say “X80” to refer to the game X = 80 and similarly for the other games.

The X-games have other important features for the experiment. Since they are very simple and fully mixed, we would not expect there to be much no-feedback learning (Weber [2003]). This is important since we are studying stochasticity in beliefs, and so want to minimize variation in beliefs due to learning. The payoffs are also “sparse” in the sense of having many payoffs set to 0. This makes the games’ structure more transparent and easier to calculate best responses. The fact that one player’s payoffs are symmetric and fixed across games also makes it easier to perceive differences across games.

For the experiment, we chose the six values of X given in Table 3. These correspond to the vertical lines in Figure 2. They were chosen so that the corresponding values of $\sigma^L$ are relatively evenly spaced on the unit interval and come close to the boundary at one end. The values of X also go well above and well below 20 so that across the set of games, one player does not always expect to receive higher payoffs. Games X80 and X5 as well as X40 and X10 are symmetric-pairs in that $\sigma^{NE,X}_{L,80} = 1 - \sigma^{NE,X}_{L,5}$ and $\sigma^{NE,X}_{L,40} = 1 - \sigma^{NE,X}_{L,10}$. This does not, however, imply the same relation for QNBE without additional conditions.\textsuperscript{16}

<table>
<thead>
<tr>
<th>X</th>
<th>80</th>
<th>40</th>
<th>10</th>
<th>5</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma^L_{NE}$</td>
<td>0.2</td>
<td>0.333</td>
<td>0.667</td>
<td>0.8</td>
<td>0.909</td>
<td>0.952</td>
</tr>
<tr>
<td>$\sigma^U_{NE}$</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

\textbf{Table 3: Selection of X-games.}

In addition to the X-games, we also included the games whose payoffs are in Appendix Table 16. D1 and D2 are dominance solvable games, which are identical up to which player faces which set of payoffs. These are included in order to derive, for each subject, a measure of strategic sophistication (using a level $k$-type framework), which we conjectured would help to rationalize observed deviations from theoretical predictions. We discuss the dominance

\textsuperscript{16}If $Q$ is \textit{scale invariant} ($Q_i(\beta v_i) = Q_i(v_i)$ for $\beta > 0$) and \textit{label invariant} ($Q_{i1}((v,w)) = Q_{i2}((w,v))$ for any payoffs $v,w \in \mathbb{R}$) and $\sigma^*$ is \textit{label invariant} ($F_i(\tilde{\sigma}_j|\sigma_j) = 1 - F_i(\tilde{\sigma}_j|1 - \sigma_j)$ for all $\sigma_j, \tilde{\sigma}_j \in (0,1)$), then $\sigma^{QNBE,X}_{L} = 1 - \sigma^{QNBE,X'}_{L}$ and $\sigma^{QNBE,X}_{U} = 1 - \sigma^{QNBE,X'}_{U}$ if $\sigma^{NE,X}_{L} = 1 - \sigma^{NE,X'}_{L}$.\textsuperscript{17}
Table 4: Games by section.

<table>
<thead>
<tr>
<th>Stage</th>
<th>Games</th>
<th>Rounds of each</th>
<th>Rounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X80, X40, X10, X5, X2, X1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>D1, D2</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>X80s</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>R1, R2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>BA</td>
<td>X80, X40, X10, X5, X2, X1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Di</td>
<td>3</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>Dj</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>X80s</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

solvable games at length in Section 6. For brevity, we will not discuss the data from the three remaining games, X80s, R1, and R2, in this paper.\textsuperscript{17}

Table 4 summarizes the games played in both stages of the experiment and the number of rounds for each. Note that, for each of the X-games, there are two rounds in the first stage and five rounds in the second stage. The dominance solvable games appeared at fixed, evenly spaced rounds.\textsuperscript{18} The other games appeared in random order subject to the same game not appearing more than once within 3 consecutive rounds. Subjects were told nothing about what games to expect, the number of times each was to appear, or their order.

3.3 Eliciting beliefs using random binary choice

We used the random binary choice (RBC) mechanism (Karni [2009]) to incentivize subjects to state their beliefs accurately.\textsuperscript{19} In an RBC, subjects are asked which option they prefer from a list of 101 binary choices, as in Table 5 with option A on the left and option B on the right. If a subject holds belief $b\%$ over the probability that event $E$ occurs and her preferences respect stochastic dominance (in particular, she does not have to be risk-
neutral), it is optimal to choose option A for questions numbered less than \( b \) and option B for questions numbered greater than \( b \). Otherwise, the subject is failing to choose the option that she believes gives the highest probability of receiving the prize.

<table>
<thead>
<tr>
<th>Question</th>
<th>Option A: $5 if the event ( E ) occurs</th>
<th>Option B: $5 with probability 0%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q.0</td>
<td>$5 if the event ( E ) occurs</td>
<td>$5 with probability 0%</td>
</tr>
<tr>
<td>Q.1</td>
<td>$5 if the event ( E ) occurs</td>
<td>$5 with probability 1%</td>
</tr>
<tr>
<td>Q.2</td>
<td>$5 if the event ( E ) occurs</td>
<td>$5 with probability 2%</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Q.99</td>
<td>$5 if the event ( E ) occurs</td>
<td>$5 with probability 99%</td>
</tr>
<tr>
<td>Q.100</td>
<td>$5 if the event ( E ) occurs</td>
<td>$5 with probability 100%</td>
</tr>
</tbody>
</table>

Table 5: Random binary choice.

Beliefs were elicited in the second stage of the experiment in which the event \( E \) was that a randomly selected subject from the first stage chose a particular action. Specifically, subjects were shown a matrix that appeared in the first stage and told that “The computer has randomly selected a round of Section 1 in which the matrix below was played.” Player 1 (blue) subjects were then asked “What do you believe is the probability that a randomly selected red player chose L in that round?”, and similarly for player 2 (red) subjects (see Appendix 11.5 for screenshots). By entering a belief into a box, a whole number between 0 and 100 inclusive, the rows of the table were filled out optimally given the stated belief (indifference broken in favor of option B). The table did not appear on subjects’ screens by default, but they could see it by “scrolling down”.

For each round selected for a subject’s belief payment, one of the 101 rows was randomly selected and she received her chosen option. If she chose option A in the selected row, a subject of the relevant type was randomly drawn and she received $5 if the randomly drawn subject chose the relevant option. If she chose option B in the selected row, she received $5 with the probability given. Since each row was selected for payment with positive probability, subjects were incentivized to state their beliefs accurately. In addition, subjects were told explicitly that it was in their best interest to state their beliefs accurately.

4 Overview of the data

Prior to testing the axioms, we examine the data at a high level. Since our procedures are novel, we benchmark our findings against those from other experiments in which data is
collected in a more conventional way. Since this paper is concerned primarily with noisy behavior, we also explore measures of variability in stated beliefs.

4.1 Actions

Throughout the paper, we refer to actions data from various parts of the experiment and in some cases pool across treatments. For clarity, we use the following notation to indicate the data source:

- \([A, \circ]\): first-stage actions, pooled across \([A,BA]\) and \([A,A]\)
- \([A,BA]\): second-stage actions from \([A,BA]\)
- \([A,BA]\): first-stage actions from \([A,BA]\)
- \([A,A]\): first-stage actions from \([A,A]\)
- \([A,A]\): second-stage actions from \([A,A]\)

We focus primarily on \([A, \circ]\) and \([A,BA]\). We consider \([A, \circ]\) because, in testing axioms on the belief-map, we must compare beliefs to the actions they refer to, and beliefs refer to the first stage. Since there is no feedback provided to subjects and the first stages are identical in \([A,BA]\) and \([A,A]\), we pool across treatments to arrive at \([A, \circ]\). We consider \([A,BA]\) because, in testing axioms on the action-map, we must associate to each belief statement a corresponding action.

Table 6 gives the empirical frequencies from \([A, \circ]\) and \([A,BA]\). We observe some differences between the two sets of frequencies. In Section 8, we show that this difference is caused by the process of belief elicitation itself and discuss the implications for our results. This does not affect our main conclusions, but requires that we be careful about what data sources we are using for different tests. In particular, we cannot pool actions data across the two stages.

Even in the first-stage, before we elicit beliefs, our procedures for collecting actions data are somewhat unusual in that we play a large number of games, without feedback, and without the same game appearing consecutively. How does our actions data compare to actions data that is collected under more standard experimental conditions? Figure 3 plots our action frequencies from \([A, \circ]\), superimposed with those from three studies, Ochs [1995], McKelvey et al. [2000], and Rutstrom and Wilcox [2009]. For inclusion, we sought studies
Table 6: Empirical action frequencies.

<table>
<thead>
<tr>
<th></th>
<th>X80</th>
<th>X40</th>
<th>X10</th>
<th>X5</th>
<th>X2</th>
<th>X1</th>
<th>D1</th>
<th>D2</th>
</tr>
</thead>
<tbody>
<tr>
<td>[A, o ]</td>
<td>(\hat{\sigma}_U)</td>
<td>0.50</td>
<td>0.42</td>
<td>0.51</td>
<td>0.40</td>
<td>0.40</td>
<td>0.31</td>
<td>0.04</td>
</tr>
<tr>
<td></td>
<td>(\hat{\sigma}_L)</td>
<td>0.27</td>
<td>0.25</td>
<td>0.66</td>
<td>0.74</td>
<td>0.74</td>
<td>0.74</td>
<td>0.78</td>
</tr>
<tr>
<td>[A, BA ]</td>
<td>(\hat{\sigma}_U)</td>
<td>0.38</td>
<td>0.39</td>
<td>0.65</td>
<td>0.61</td>
<td>0.51</td>
<td>0.49</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>(\hat{\sigma}_L)</td>
<td>0.21</td>
<td>0.22</td>
<td>0.74</td>
<td>0.78</td>
<td>0.83</td>
<td>0.82</td>
<td>0.80</td>
</tr>
</tbody>
</table>

Figure 3: Actions data. This figure plots the first stage empirical frequencies \([A, o]\) with 90% confidence bands (clustered by subject), superimposed with the empirical frequencies from other studies.

That played games with “sparse” payoffs\(^{20}\) and \(\sigma_{L}^{NE} = \frac{1}{2}\) (after relabelling). This latter feature allows us to plot their data in our figure as a function of \(\sigma_{L}^{NE}\). In these studies, a single game was played 36-50 times consecutively with feedback against either randomly re-matched opponents or a fixed opponent. We find that our data is remarkably close to theirs despite the differences in procedures.

That being said, we cannot find precedents in the literature for games closely matching our more symmetric games—those with \(\sigma_{L}^{NE}\) relatively close to \(\frac{1}{2}\). The only surprising behavior is for \(X40\) in which the data falls significantly outside of the QRE-NBE region. In all cases, however, the empirical frequencies from individual games can be supported as QNBE outcomes, as we show in Appendix Figure 26.

\(^{20}\)Goeree and Holt [2001] played similar games one-shot without sparse payoffs and found data that deviated much farther from NE than in any of these other papers.
4.2 Rates of best response

As we did with the actions data, we compare our findings on beliefs to a benchmark from the literature on belief elicitation. To this end, we look at rates of best response to stated beliefs, which this literature has suggested as a method for validating elicited beliefs (see Schotter and Trevino [2014] for a discussion of this view). Appendix Figure 27 plots histograms of subjects’ rates of best response from the X-games. Compared to the study of Nyarko and Schotter [2002] who report an average rate of 75% for an asymmetric matching pennies game played many times with feedback, we find lower rates for player 1 (64%) and higher rates for player 2 (85%).

Appendix Table 15 shows the average rates of best response for each game. Our relatively low rates for player 1 are driven by the very asymmetric games with low values of X. For games with higher values of X that resemble the games from Nyarko and Schotter [2002] more closely, we have very similar rates. That our rates are higher for player 2 is unsurprising since player 2 faces symmetric payoffs and thus has an easier choice to make for any given belief.

4.3 Are beliefs noisy?

To our knowledge, this is the first study to have multiple belief elicitation per subject-game without feedback. A natural question is: are beliefs noisy?

For each subject and X-game, we calculate the spread of her beliefs—the highest belief minus the lowest—across the five belief statements. We average this across the six X-games for each subject to get an average spread measure. Figure 4 plots histograms of subjects’ spreads by player role. There is considerable heterogeneity in spreads, and there is a right tail of very noisy subjects. The average spreads are 24 and 21 belief-points for player 1- and player 2-subjects, respectively. This seems large to us, though these are lower than that expected of the benchmark of uniformly randomizing over a range of 50 belief points (expected value of 33). In unreported results, ANOVA reveals much larger between- than within-subject variance in beliefs for all games and roles. This suggests that, while subjects do have noisy beliefs, patterns in individual subjects’ beliefs are relatively stable but heterogeneous.

\footnote{We consider this a natural benchmark because it results from believing one action is more likely than another but otherwise reporting beliefs randomly subject to that constraint.}
Figure 4: Subjects’ spreads of beliefs. This figure gives histograms of subjects’ spreads of beliefs, averaged across all X-games.

4.4 Actions given expected payoffs (given beliefs)

The premise of quantal response is that beliefs determine actions only insofar as they pin down expected payoff vectors. Hence, we visualize the variation in expected payoffs observed in the data and the extent to which it is predictive of the actions subjects take.

The left panel of Figure 5 plots the convex hull of all expected payoff vectors that we may observe in the data. \((v_{i1}, v_{i2})\) is a vector of expected payoffs. In the case of player 1, \(v_{i1}\) and \(v_{i2}\) are the payoffs to \(U\) and \(D\), respectively. In the case of player 2, \(v_{i1}\) and \(v_{i2}\) are the payoffs to \(L\) and \(R\), respectively. Each of the straight black lines refer to expected payoffs given beliefs that can be observed in different player-game combinations. The line labelled “\(X_1\)” refers to player 1 in \(X_1\), the line labelled “\(X_2\)” refers to player 1 in \(X_2\), and similarly for lines labelled “\(X_5\), “\(X_10\), “\(X_40\),” and “\(X_80\). Recalling that player 2’s payoff matrix is fixed across games, the line labelled “\(P_2\)” refers to player 2 in any of these games. The right panel plots, as black dots, the empirical expected payoff vectors (i.e. given stated beliefs) and associated actions, where \(U\) and \(L\) are coded as 1 and \(D\) and \(R\) are coded as 0. We also plot a surface that gives the expected action as a function of payoff vectors based on a local linear (loess) regression. The left panel gives the associated level sets.

From this exercise, we conclude that there is a wide range of belief statements—and thus of expected payoff vectors—both within and across games. Furthermore, this variation is predictive of the actions subjects take.
Figure 5: *Actions given expected payoffs (given beliefs).* The left panel gives the convex hull of all expected payoff vectors that may be observed in the data in any of the $X$-games. $(v_{i1}, v_{i2})$ refers to either the payoffs to $(U, D)$ for player 1 or the payoffs to $(L, R)$ for player 2. “X1”, “X2”, ..., and “X80” refer to player 1’s vectors in the corresponding games, and “P2” refers to player 2’s vectors in any of the games. The right panel plots the action taken as a function of the expected payoff vectors observed in the data, with $U$ and $L$ coded as 1 ($D$ and $R$ coded as 0). The surface is the predicted action from a local linear (lowess) regression (smoothing parameter 0.85). The left panel gives the corresponding level sets.

5 Testing the Axioms

We test each axiom by formulating a statistical hypothesis test with the axiom as the null hypothesis.

5.1 Responsiveness

*Responsiveness* states that an all-else-equal increase in the expected payoff to some action increases the probability that action is played. To test this, we must associate actions with their expected payoffs given beliefs, and so we use the data from [A,BA].

Since player 1’s payoff parameter $X$ is different in each game and there is variation in beliefs across games, not all of player 1’s expected payoff vectors across games can be ordered by an all-else-equal increase in the payoff to some action. In such cases, *responsiveness* imposes no restrictions on stochastic choice. With additional conditions, one can complete the order, but we do not pursue that here.\(^{22}\) Instead, we first consider tests game-by-game.

\(^{22}\)Consider two unordered vectors, $v_i = (v_{i1}, v_{i2}) = (5, 2)$ and $w_i = (w_{i1}, w_{i2}) = (3, 1)$. One can complete the order with additional restrictions. For instance, if the quantal response function is translation
Then, we consider player 2 only, whose payoff parameters are fixed across games, allowing us to pool data across the entire set of games. In all cases, the variation in expected payoffs is through variation in beliefs.

Since expected payoffs are one-to-one with beliefs within a game, responsiveness is easily translated into a condition on beliefs. For player 1 and game $x$, we state the hypothesis\textsuperscript{23} as

$$H_0 : \hat{Q}_U(\tilde{u}_1^x(\sigma_L')) \text{ is everywhere weakly increasing in } \sigma_L'.$$

Similarly, for player 2:

$$H_0 : \hat{Q}_L(\tilde{u}_2^x(\sigma_U')) \text{ is everywhere weakly decreasing in } \sigma_U'.$$

We visualize the relevant data in Figure 6, which plots estimates of $\hat{Q}$ for games X80 and X5 for both players. Appendix Figure 28 gives the plots for all six games. These are simply the predicted action frequencies from regressing actions on beliefs using a flexible specification (see figure caption for details). Recall that, for each game and player role, there are five observations per subject and so these plots represent a mix of between- and within-subject variation. The vertical dashed line gives the indifferent belief $\sigma_j' = \sigma_j^{NE}$ and the horizontal dashed line is set to one-half.

To get a better sense of the raw data, the plots also include belief histograms and the average action within each of ten equally spaced bins (black dots). In some of these bins, there are very few datapoints and so the average action is not very meaningful. The predicted $\hat{Q}$ uses data much more efficiently.

Responsiveness is equivalent to an increasing slope for player 1 and a decreasing slope for player 2. Applying the non-parametric monotonicity test of Bowman et al. [1998] (see Appendix 11.3 for details of implementation),\textsuperscript{24} we reject this for both players in all six games with $p$-values close to 0. However, we must be careful in interpreting this result. Different subjects form different beliefs, and hence the $\hat{Q}$-curves plotted in Figure 6 are invariant, then $Q_{i1}(v_i) > Q_{i1}(w_i)$ since $Q_{i1}((5, 2)) = Q_{i1}((4, 1)) > Q_{i1}((3, 1))$ where the inequality is due to responsiveness. If the quantal response function is scale invariant, then $Q_{i1}(w_i) > Q_{i1}(v_i)$ since $Q_{i1}(3, 1) = Q_{i1}(6, 2) > Q_{i1}(5, 2)$ where the inequality is due to responsiveness.

\textsuperscript{23}The null hypothesis, by allowing for weak monotonicity, is slightly weaker than the axiom, but it allows for the use of more standard tests.

\textsuperscript{24}It is a bootstrap-based test where the data generating process is, heuristically, the best-fit (non-parametrically estimated) monotonic (upward sloping for player 1, downward sloping for player 2) function plus noise, and the $p$-value is constructed as the fraction of simulations for which a non-parametric regression estimator is non-monotonic.
Figure 6: Action frequencies predicted by beliefs. For each player and games X80 and X5, we plot the predicted values (with 90% error bands) from restricted cubic spline regressions of actions on beliefs (4 knots at belief quintiles, standard errors clustered by subject). Belief histograms appear in gray and the average action within each of ten equally spaced bins appear as black dots. The vertical dashed line is the indifferent belief $\sigma_j' = \sigma_j^{NE}$, and the horizontal line is set to one-half.
patched together from different subjects representing different parts of the domain. Hence, the violation could result from individual subjects who violate responsiveness to variations in their own beliefs or it could be, in the case of player 1 (and similarly for player 2), there are subjects who tend to hold lower beliefs and favor taking $U$ (whose payoff increases in beliefs). This latter possibility could lead to violations of responsiveness even if all individual subjects are responsive to variations over the range of their own stated beliefs.

To determine if individual subjects are responsive to variations in their own stated beliefs, we run fixed effects regressions for different regions of stated beliefs (responsiveness is a local property, so we wish to maintain some flexibility in the specification). Let \($a_{ix}, b_{ix}\) be the \(l\)th action-belief pair of subject \(s\) in role \(i\) in game \(x\). As has been our convention, the actions of \(U\) and \(L\) are coded as 1, and \(D\) and \(R\) are coded as 0 (e.g. \(a_{ix} = 1\) if player 1 takes \(U\)). Let \(\bar{a}_{ix} = \frac{1}{5}\sum_i a_{ix}\) and \(\bar{b}_{ix} = \frac{1}{5}\sum_i b_{ix}\) be the subject-level averages. For each role \(i\) and game \(x\), we run the following regression for each tercile of belief statements \(\{b_{ix}\}_{ix}\), which we label as “low”, “medium”, and “high” beliefs:\(^{25}\)

\[
a_{ix} - \bar{a}_{ix} = \beta(b_{ix} - \bar{b}_{ix}) + \epsilon_{ix}. \tag{3}
\]

Since there is no difference across subjects in the averages of their demeaned variables (by construction), the coefficient estimate \(\hat{\beta}\) reflects within-subject variation.

The results are displayed in Table 7. Consistent with responsiveness, we find that every slope is positive for player 1 and all but one (which is extremely close to 0 and insignificant) are negative for player 2, with many of these being highly statistically significant. Furthermore, the magnitudes are large: a majority of slopes have an absolute value greater than 0.005,\(^{26}\) indicating that a 1 percentage point change in belief is associated with a 0.5 percentage point change in the probability of taking an action. Since the slopes all have the sign predicted by responsiveness, this suggests that individual subjects are overwhelmingly responsive.

Following our discussion at the beginning this section, we now turn to player 2 data pooled across all games. Using this data, the top panel of Figure 7 reproduces Figure 6, and Appendix Table 17 presents results of the fixed effects regressions. Since we have much more data that is distributed more uniformly within the space of possible beliefs, we run regressions for each belief-quintile instead of tercile (first column) and also present a version

\(^{25}\)Results are largely unchanged, but a bit underpowered, if instead use 4 or 5 bins.

\(^{26}\)For player 1, the absolute slopes average 0.065 and range from 0.000-0.015. For player 2, the average is 0.065 and range from 0.000-0.018.
Table 7: *Fixed effect regressions of actions on beliefs.* For each game and player, we divide individual belief statements into terciles—low, medium, and high beliefs. For each belief tercile, we run a separate linear regression of actions on beliefs that are both first demeaned by subtracting subject-specific averages. Standard errors are clustered by subject.

<table>
<thead>
<tr>
<th>Player 1</th>
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<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
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<td></td>
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<td>X10</td>
<td>X5</td>
<td>X2</td>
<td>X1</td>
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<td>0.008**</td>
<td>0.010***</td>
<td>0.005**</td>
<td>0.004**</td>
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<tr>
<td></td>
<td>(0.958)</td>
<td>(0.077)</td>
<td>(0.017)</td>
<td>(0.002)</td>
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<td>(0.043)</td>
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<tr>
<td>medium beliefs</td>
<td>0.007**</td>
<td>0.010**</td>
<td>0.015***</td>
<td>0.005</td>
<td>0.006</td>
<td>0.006*</td>
</tr>
<tr>
<td></td>
<td>(0.033)</td>
<td>(0.020)</td>
<td>(0.000)</td>
<td>(0.153)</td>
<td>(0.141)</td>
<td>(0.051)</td>
</tr>
<tr>
<td>high beliefs</td>
<td>0.005*</td>
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<td>0.004</td>
<td>0.005</td>
<td>0.004</td>
<td>0.007***</td>
</tr>
<tr>
<td></td>
<td>(0.052)</td>
<td>(0.002)</td>
<td>(0.448)</td>
<td>(0.164)</td>
<td>(0.283)</td>
<td>(0.004)</td>
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*p*-values in parentheses

* p < .1, ** p < .05, *** p < .01

<table>
<thead>
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<td>X10</td>
<td>X5</td>
<td>X2</td>
<td>X1</td>
</tr>
<tr>
<td>low beliefs</td>
<td>-0.010***</td>
<td>-0.018***</td>
<td>-0.005**</td>
<td>-0.007*</td>
<td>-0.004*</td>
<td>-0.004</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.000)</td>
<td>(0.038)</td>
<td>(0.074)</td>
<td>(0.071)</td>
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<tr>
<td>medium beliefs</td>
<td>-0.013***</td>
<td>-0.006</td>
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<td>-0.010*</td>
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<tr>
<td></td>
<td>(0.000)</td>
<td>(0.174)</td>
<td>(0.490)</td>
<td>(0.930)</td>
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<tr>
<td>high beliefs</td>
<td>-0.008**</td>
<td>-0.004</td>
<td>-0.010***</td>
<td>-0.007**</td>
<td>-0.004</td>
<td>-0.005***</td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
<td>(0.199)</td>
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<tr>
<td>Observations</td>
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<td>280</td>
<td>280</td>
<td>280</td>
</tr>
</tbody>
</table>

*p*-values in parentheses

* p < .1, ** p < .05, *** p < .01
Figure 7: Player 2 subjects–actions and beliefs, pooled across games. All plots involve player 2-subjects whose data is pooled across all games. Action $L$ is coded as 1, and action $R$ is coded as 0. The top panel uses all player 2-subjects and gives the predicted action frequencies (with 90% error bands) from restricted cubic spline regressions of actions on beliefs (4 knots at belief quintiles, std. errors clustered by subject) superimposed over the histogram of beliefs. The remaining plots are for specific player 2 subjects. The solid black curve is the parametric regression estimator used in Step 2 of the Bowman et al. [1998] test (Appendix 11.3), and data is separately marked for each game. All datapoints involve a value of 1 or 0 on the vertical axis, but are plotted with a bit of (vertical) noise for visual clarity.
using evenly spaced bins of 20 belief points (second column). Consistent with responsiveness, we find that every slope is highly statistically negative and with large magnitudes.

An interesting question is whether within-subject variation in beliefs has predictive power only insofar as beliefs go on one side or the other of the indifferent belief. Inspecting Appendix Table 17, the answer is definitive. Even for player 2, whose indifferent belief is salient, constant across games, and invariant to curvature in the utility function, this variation is highly predictive of actions. Restricting attention to beliefs that are in the bottom or top quintiles—at least 30 points away from the indifferent belief—a 1 percentage point change in belief is associated with a 0.5-0.6 percentage point change in the probability of taking an action.

Being able to pool across games for player 2-subjects results in many more datapoints per subject (30 as opposed to 5) that typically cover much more of the space of possible beliefs. In particular, this allows us to test for responsiveness for individual subjects using the Bowman et al. [1998] test. Figure 7 plots some representative individual subjects’ data pooled across all six games, superimposed by the non-parametric regression estimator used in Step 2 of the Bowman et al. [1998] test (see Appendix 11.3). The four subjects depicted in the figure are representative of the types of subjects we observe: subject 65 is characterized by step function-like responsiveness and always best responds; subject 87 is also responsive, but has action-probabilities that are more continuous in beliefs; subject 59 is similar to subject 87 but noisier, and the non-parametric test rejects responsiveness; subject 82 is an “opposite type” who fails responsiveness trivially. In all, responsiveness is rejected in only 19% of player 2-subjects.

5.2 Monotonicity

Monotonicity is a weakening of best response which states that, given beliefs, the action with a higher expected payoff is played more often than not and, if players are indifferent, they uniformly randomize. Since we must associate expected payoffs given beliefs to actions, we again use the data from [A,BA].

For the games studied in this paper, since players are indifferent when their beliefs equal the opponent’s Nash equilibrium strategy, monotonicity takes a particularly simple form. For player 1 and game x, we state the hypothesis as

\[ \text{subject 65 is characterized by step function-like responsiveness and always best responds; subject 87 is also responsive, but has action-probabilities that are more continuous in beliefs; subject 59 is similar to subject 87 but noisier, and the non-parametric test rejects responsiveness; subject 82 is an “opposite type” who fails responsiveness trivially. In all, responsiveness is rejected in only 19% of player 2-subjects.} \]

\[ \text{Monotonicity is a weakening of best response which states that, given beliefs, the action} \]

\[ \text{with a higher expected payoff is played more often than not and, if players are indifferent, they uniformly randomize. Since we must associate expected payoffs given beliefs to actions, we again use the data from [A,BA].} \]

\[ \text{For the games studied in this paper, since players are indifferent when their beliefs equal the opponent’s Nash equilibrium strategy, monotonicity takes a particularly simple form. For player 1 and game x, we state the hypothesis as} \]

\[ \text{Many subjects look similar to subject 65 except with up to 2 “mistakes”, which typically does not lead to a rejection of responsiveness.} \]

\[ \text{30} \]
\[ H_0 : Q_U(\bar{u}_1^x(\sigma'_L)) \geq \frac{1}{2} \text{ if and only if } \sigma'_L \geq \sigma^{NE,x}_L. \]

Similarly, for player 2 and game \( x \):

\[ H_0 : Q_L(\bar{u}_2^x(\sigma'_U)) \geq \frac{1}{2} \text{ if and only if } \sigma'_U \geq \sigma^{NE,x}_U. \]

In order to visualize potential monotonocity violations, we appeal once again to Figure 6, which plots estimates of \( \hat{Q} \) for games X80 and X5 for both players (see figure caption for details; see Appendix Figure 28 for all six games). The vertical dashed line gives the indifferent belief \( \sigma'_j = \sigma^{NE}_j \) and the horizontal dashed line is set to one-half. As opposed to responsiveness that concerns the slope, monotonocity concerns the levels of the graph. Specifically, for player 1 (left panels), \( \hat{Q}_U \) should be less than \( \frac{1}{2} \) to the left of the vertical line and greater than \( \frac{1}{2} \) to the right of the vertical line; for player 2 (right panels), \( \hat{Q}_L \) should be greater than \( \frac{1}{2} \) to the left of the vertical line and less than \( \frac{1}{2} \) to the right of the vertical line.

In testing monotonocity, we conduct the analysis at the aggregate level since we have only 5 belief statements for each subject-game. Unlike for responsiveness, there is no issue in aggregation. Since monotonocity is a condition that holds pointwise, if all subjects have monotonic quantal response over the range of their stated beliefs (even if different subjects form very different beliefs), the aggregate will also be monotonic.

Our test for monotonocity is the natural one suggested by eyeballing Figure 6. After running flexible regressions of actions on beliefs, we calculate the standard error of the prediction (clustering by subject), which we use to calculate error bands for the estimated \( \hat{Q} \). From the figure, one can observe rejections of the null at the given level of significance. For instance, in the top left panel (game X80, player 1), we see that for beliefs just above 20, whereas monotonocity requires that \( Q \) should be above \( \frac{1}{2} \), we observe that the estimated \( \hat{Q} \) is significantly below \( \frac{1}{2} \). Since it is the 90\% error band that is plotted, inspection reveals that monotonocity is rejected with a \( p \)-value less than 0.1. Similarly, if the 95\% error band still leads to a violation, then the \( p \)-value is less than 0.05. By considering error bands of increasing size, all violations will eventually disappear. Hence, we calculate the \( p \)-value as \( c \), where the 100(1 – c)\% error band is the smallest which results in no violations.

One weakness of the test is that it is sensitive to the regression specification, so we report the results (\( p \)-values) of the statistical tests in Table 8 for 5 different specifications (see table caption for details). The second panel of the table gives a reduced-form measure of the degree of monotonocity violations—the total area enclosed between \( \hat{Q} \) and the one-half line.
### Tests of Monotonicity ($p$-values)

<table>
<thead>
<tr>
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<th>Player 1 ($Q_U$)</th>
<th>Player 2 ($Q_L$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C^*_4$</td>
<td>$L^*_4$</td>
</tr>
<tr>
<td>X80</td>
<td>0.00***</td>
<td>0.02**</td>
</tr>
<tr>
<td>X40</td>
<td>0.01***</td>
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<td>X10</td>
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<td>X5</td>
<td>0.00***</td>
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</tr>
<tr>
<td>Avg</td>
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<td>0.00***</td>
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</table>

### Size of Monotonicity Violation

<table>
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<th>Player 1 ($Q_U$)</th>
<th>Player 2 ($Q_L$)</th>
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<tbody>
<tr>
<td></td>
<td>$C^*_4$</td>
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</tr>
<tr>
<td>X80</td>
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<td>X40</td>
<td>0.82</td>
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<tr>
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</tr>
<tr>
<td>Avg</td>
<td>3.81</td>
<td>4.12</td>
</tr>
</tbody>
</table>

Table 8: Testing monotonicity. For each player and game, we test for monotonicity in the manner described in Section 5.2 using 5 different regression models to estimate $\hat{Q}$. The 5 models are based on restricted splines: cubic with 4 knots based on belief quintiles ($C^*_4$); linear with 4 knots based on belief quintiles ($L^*_4$); and cubic with 5, 6, or 7 equally spaced knots ($C^*_5$, $C^*_6$, and $C^*_7$, respectively). The top panel reports $p$-values, as well as the $p$-values averaged across games for a given model and averaged across models for a given game. The bottom panel reports a reduced-form measure of monotonicity violations—the total area enclosed between $\hat{Q}$ and the one-half line over beliefs that lead to (not necessarily significant) violations.
over beliefs that lead to (not necessarily significant) violations.

We find that monotonicity cannot be rejected for player 2 in any game, and this is consistent across regression specifications. In particular, it is not rejected with very high \( p \)-values for the most flexible specifications (see table caption for details). For player 1, on the other hand, we observe consistent and highly significant violations of monotonicity in all games that occur over a region of 5-30 belief points, depending on the game. Moreover, based on the belief histograms in Figure 6, it is clear that a large mass of beliefs (including the mode) fall in the regions with monotonicity violations.

From Figure 6, it is clear that the nature of player 1’s monotonicity violations is systematic. For \( X > 20 \), the violations occur over an interval of beliefs just “right of” the indifferent belief, and for \( X < 20 \), the violations are over an interval of beliefs just “left of” the indifferent belief. We consider explanations for this pattern in Section 6.1.

A weak implication of monotonicity is that best responses will be taken with probability greater than one-half. As shown previously in Appendix Table 15, best responses are taken with probability greater than one-half in all games. Thus, even though subjects tend to best respond to the beliefs that they form, they systematically fail to best respond to beliefs that realize in particular regions of the belief-space. Hence, our analysis expands upon previous studies using elicited beliefs (e.g. Costa-Gomes and Weizsacker [2008] and Rey-Biel [2009]) that have focused only on rates of best response.

5.3 Belief-responsiveness

Belief-responsiveness states that, if the frequency of player \( j \)’s action increases, so too does the distribution of player \( i \)’s beliefs in the sense of first-order stochastic dominance. Recalling that the beliefs are elicited about behavior in the first stage and that the first stages are identical across the treatments, we use the beliefs data from \([A, BA]\) and the actions data from \([A, \circ]\).

Across games \( x \) and \( y \), we seek tests of the form\(^\text{28}\)

\[
H_0 : \sigma^x_j > \sigma^y_j \text{ and } F_i(\cdot | \sigma^x_j) FOSD F_i(\cdot | \sigma^y_j), \quad \text{or}
\]

\[
\sigma^y_j > \sigma^x_j \text{ and } F_i(\cdot | \sigma^y_j) FOSD F_i(\cdot | \sigma^x_j). \tag{4}
\]

Prior to testing, we visualize the data in Figure 9, which plots histograms of stated beliefs, superimposed with median beliefs (solid vertical lines) and the corresponding empirical

\(^{28}\)The null hypothesis here is slightly weaker than used in the axiom, but it allows for the use of more standard tests.
frequencies of actions (dashed vertical lines). It appears that the distributions of beliefs shift monotonically in $X$ in the direction predicted by QNBE: as $X$ increases, player 2 believes that player 1 will play $U$ more often and player 1 believes player 2 will play $L$ less often. Furthermore, plotting the CDFs of beliefs in Figure 8 suggests that the belief distributions are ordered by stochastic dominance. The empirical action frequencies also typically, but not always, move in the same direction, consistent with belief-responsiveness.

Our test of hypothesis (4) is simple and conservative in the sense of not over-rejecting. To this end, we perform one-sided tests of the null hypotheses $H_0 : \sigma^x_j > \sigma^y_j$ and $H_0 : F_i(\cdot | \sigma^y_j) \succeq_{FOSD} F_i(\cdot | \sigma^x_j)$. If $\sigma^y_j$ is greater than $\sigma^x_j$, belief-responsiveness dictates that $F_i(\cdot | \sigma^y_j) \succeq_{FOSD} F_i(\cdot | \sigma^x_j)$, so we say that there is a failure of the axiom if we reject both that $\sigma^x_j > \sigma^y_j$ and that $F_i(\cdot | \sigma^y_j) \succeq_{FOSD} F_i(\cdot | \sigma^x_j)$.\footnote{This would be a conservative test if a situation in which belief distributions were unordered by stochastic dominance did not lead to rejection of the axiom, and in most cases the ordering is clear (Figure 8).}

Table 9 reports the $p$-values of the one-sided tests of $H_0 : \sigma^x_j > \sigma^y_j$ and $H_0 : F_i(\cdot | \sigma^y_j) \succeq_{FOSD} F_i(\cdot | \sigma^x_j)$ for every combination of games $x$ and $y \neq x$ and for each player $i$ and her opponent $j$ (see table caption for details of these tests). These are reported in matrix form as entries in row $x$ and column $y$. We find only one significant violation across the many comparisons. This can be seen from the $p$-values in bold, indicating rejections of both $\sigma^X_{40} > \sigma^X_{10}$ and $F_1(\cdot | \sigma^X_{10}) \succeq_{FOSD} F_1(\cdot | \sigma^X_{40})$. We conclude that belief-responsiveness cannot be rejected in our data.

![Figure 8: CDFs of belief distributions. We plot the empirical CDFs of belief distributions. The left panel is for player 2’s beliefs about $U$, and the right panel is for player 1’s beliefs about $L$.](image-url)
Figure 9: Belief distributions. The left panel is for player 2’s beliefs about $U$, and the right panel is for player 1’s beliefs about $L$. The solid lines mark the median of $i$’s beliefs and the dashed line marks the empirical frequency of $j$’s actions.
### Table 9: Testing belief-responsiveness

The top panel reports $p$-values from tests of $H_0: \sigma^x_j > \sigma^y_j$ across games $x$ (row) and $y$ (column). This is from standard $t$-tests, clustering by subject. The bottom panel reports $p$-values from tests of $H_0: F_i(\cdot|\sigma^x_j) \succeq_{FOSD} F_i(\cdot|\sigma^y_j)$ across games $x$ (row) and $y$ (column). This is from non-parametric Kolmogorov-Smirnov-type tests in which the test statistic is bootstrapped following Abadie [2002] in a way that preserves the within-subject correlation of beliefs observed in the data (see Appendix 11.3 for details). The entries in bold correspond to the only rejection: $\sigma^x_1 \not\succeq_{FOSD} \sigma^y_1$ and $F_i(\cdot|\sigma^x_1) \not\succeq_{FOSD} F_i(\cdot|\sigma^y_1)$. 

<table>
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<tr>
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<th>Players $i = 2, j = 1$ ($p$-values)</th>
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<table>
<thead>
<tr>
<th>Beliefs</th>
<th>Players $i = 1, j = 2$ ($p$-values)</th>
<th>Players $i = 2, j = 1$ ($p$-values)</th>
</tr>
</thead>
<tbody>
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</tr>
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<tr>
<td></td>
<td>$X_{1}$</td>
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</tbody>
</table>
5.4 Unbiasedness

Unbiasedness states that beliefs are correct on median. Hence, for each player $j$ and game $x$, we test the hypothesis

$$H_0: \text{med}(\sigma_j^{*,x}) = \sigma_j^x,$$

where $\text{med}(\cdot)$ denotes the median of a random variable. Once again, we use the beliefs data from $[A,BA]$ and the actions data from $[A, \circ]$.

![Figure 10: Bias in beliefs. The left panel gives player 1’s action frequency from $[A, \circ]$ and the median of player 2’s beliefs about player 1. Blue circles are individual belief statements. The right panel gives player 2’s action frequency from $[A, \circ]$ and the median of player 1’s beliefs about player 2. Red circles are individual belief statements.](image)

Unbiasedness requires that beliefs are correct on median, so we plot in Figure 10 the aggregate action frequencies and median beliefs as well as the individual belief statements. Appendix Table 18 reports the bias in both median- and mean-beliefs with $p$-values of the hypothesis that beliefs are unbiased (see caption for details).

We find that player 1’s beliefs about player 2 ($\hat{\sigma}_L^*$) are remarkably unbiased in that we fail to reject unbiasedness for most games individually. When using the mean belief instead of median, we find that there is a small “conservative” bias in the sense that mean beliefs are closer to the uniform distribution than the actual distribution of actions. Such bias has been documented by Huck and Weizsacker [2002] and Costa-Gomes and Weizsacker [2008] in other settings, and is relatively common in experiments in which beliefs are elicited.
More interestingly, we find that player 2’s beliefs about player 1 ($\hat{U}$) are very “extreme” and we reject unbiasedness for all games (and similarly for mean-unbiasedness). Whereas player 1’s actions are relatively close to uniform for all values of $X$, player 2 overwhelmingly believes player 1 takes $U$ when $X$ is large and similarly takes $D$ when $X$ is small. Hence, the nature of bias depends qualitatively on player role, which we take as one of the key facts to be explained in the next section.

6 Explaining the Failure of the Axioms

Having identified violations of monotonicity and unbiasedness, we offer behavioral explanations.

6.1 Monotonicity

We observe, for player 1 only, a systematic failure of monotonicity, as shown in Figure 6. Under the maintained assumption of expected utility, utility is linear in matrix payoffs since they are in probability points. Monotonicity thus predicts that $\hat{Q}_U$ should cross the one-half line at the indifferent belief plotted in the figure (dashed vertical lines). However, we observe systematic failures: for $X > 20$, $\hat{Q}_U$ crosses to the right of the indifferent belief, and for $X < 20$, $\hat{Q}_U$ crosses to the left of the indifferent belief. If, however, there is non-linearity in the utility function over matrix payoffs, the actual indifferent belief—and thus where monotonicity predicts $\hat{Q}_U$ crosses the one-half line—may deviate systematically from that under linear utility. The proposition below states that, with concavity in the utility function, the indifferent belief moves right for $X > 20$ and left for $X < 20$, which is consistent with the observed violations.

**Proposition 7.** Let $w$ and $v$ be any strictly increasing Bernoulli utility functions. For player 1 in game $X$, $w$ induces expected utility vectors $\bar{w}^X = (\bar{w}_U^X, \bar{w}_D^X) : [0, 1] \rightarrow \mathbb{R}^2$. Let $\bar{\sigma}^{w,X}_L$ be the unique indifferent belief such that $\bar{w}_U^X(\bar{\sigma}^{w,X}_L) = \bar{w}_D^X(\bar{\sigma}^{w,X}_L)$. (i) If $w$ is more concave than $v$ ($w = f(v)$ for $f$ concave), then $\bar{\sigma}^{w,X}_L > \bar{\sigma}^{v,X}_L$ for $X > 20$ and $\bar{\sigma}^{w,X}_L < \bar{\sigma}^{v,X}_L$ for $X < 20$. (ii) if $w$ is concave, then $\bar{\sigma}^{w,X}_L \in (\sigma^{NE,X}_L, \frac{1}{2})$ for $X > 20$ and $\bar{\sigma}^{w,X}_L \in (\frac{1}{2}, \sigma^{NE,X}_L)$ for $X < 20$.

**Proof.** See Appendix 11.2.

Since player 2 faces symmetric payoffs, curvature does not affect her indifferent belief. Hence, both players having concave utility is qualitatively consistent with the whole of the data.
To test for concavity, we fit a random utility model (e.g. Luce [1959]) with curvature to each player 1-subject’s actions data given belief statements. Since we will fit random utility models to both players’ data later on, we keep the notation general by using $i$ for player role. The data of subject $s$ in role $i$ is a set of 30 action-belief pairs \{\hat{a}^i_{sl}, \hat{b}^i_{sl}\}_{lX}$ where $l \in \{1, \ldots, 5\}$ indexes each elicitation and $X$ indexes the game. We assume that the utility function is the constant relative risk aversion (CRRA) utility function with curvature parameter $\rho$, which has been modified to allow for 0 monetary payoffs by adding a constant $\epsilon > 0$ (arbitrarily pre-set to 0.001) to each payoff. We also normalized utility so that it is between 0 and 1:

$$w(z; \rho) = \frac{(z + \epsilon)^{1-\rho} - \epsilon^{1-\rho}}{(80 + \epsilon)^{1-\rho} - \epsilon^{1-\rho}}.$$ 

This utility function induces, for each game $X$ and stated belief $\hat{b}^i_{sl}$, a vector of expected utilities $\bar{w}^X_i(\hat{b}^i_{sl}; \rho) = (\bar{w}^X_{i1}(\hat{b}^i_{sl}; \rho), \bar{w}^X_{i2}(\hat{b}^i_{sl}; \rho))$. We assume that the probability of taking the first action ($U$ in the case of player 1, $L$ in the case of player 2) depends only on this vector, based on the Luce quantal response function with sensitivity parameter $\mu_a > 0$:

$$p_X(\hat{a}^i_{sl} | \hat{b}^i_{sl}; \rho, \mu_a) = \frac{\bar{w}^X_{i1}(\hat{b}^i_{sl}; \rho)^{\frac{1}{\mu_a}}}{\bar{w}^X_{i1}(\hat{b}^i_{sl}; \rho)^{\frac{1}{\mu_a}} + \bar{w}^X_{i2}(\hat{b}^i_{sl}; \rho)^{\frac{1}{\mu_a}}}.$$ 

For subject $s$ in role $i$, we choose $\rho$ and $\mu_a$ to maximize the log-likelihood of observed actions given stated beliefs:

$$L^s(\hat{a}; \rho, \mu_a) = \sum_{X} \sum_{l=1}^{5} \ln(p_X(\hat{a}^i_{sl} | \hat{b}^i_{sl}; \rho, \mu_a)).$$

We find that for 37 of 54 player 1-subjects (69%), a likelihood ratio test rejects the restriction of linear utility, that $\rho = 0$, at the 5% level. For 31 of those 37 subjects (84%), the estimated $\hat{\rho}$ is positive, indicating concavity.

We also fit $\rho$ and $\mu_a$ to the player 1 data pooled across all subjects and games, i.e. to maximize $L(\hat{a}; \hat{b}^i_{sl}; \rho, \mu_a) = \sum_s L^s(\hat{a}; \hat{b}^i_{sl}; \rho, \mu_a)$. We find the estimate $\hat{\rho} = 0.87$, indicating concavity. In Figure 11, we reproduce Figure 6 for player 1, but we now plot the indifferent

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30 By construction, $w(0; \rho) = 0$ and $w(80; \rho) = 1$.

31 The Luce rule (5) fits the data much better than the logit quantal response function, but is undefined when one of the expected utilities is 0. This happens if and only if the stated belief is 0 or 100, which occurs very few times in the data. When this occurs, we instead use 1 or 99, respectively, to calculate the expectations.
beliefs implied by the best-fit utility function as solid vertical lines (see Appendix Figure 29 for all six games). Each such line intersects \( \hat{Q}_U \) near to where it crosses the horizontal one-half line. Hence, if the subjects admitted a representative agent with this concave utility, nearly all of the monotonicity violations would disappear. This also captures the fact that the regions of violations are larger for the more asymmetric games (compare, for example, X10 and X1 in Appendix Figure 29).

There are several potential explanations as to why subjects’ behavior can be rationalized by concave utility over matrix payoffs. First, it could be that subjects thought of probability points as money and were risk averse. We do not believe, however, that subjects were confused about the nature of payoffs: they had to answer four questions demonstrating understanding of how to map players’ actions in a game to payoff outcomes (see Section 3), and these emphasized that payoffs were in probability points. Second, it could be that subjects simply wanted to “win” the game by taking the action that maximized the probability of earning positive probability points unless the other action was sufficiently attractive.

6.2 Unbiasedness

Unbiasedness is rejected in the X-games: player 1 forms unbiased/conservative beliefs whereas player 2 forms extreme beliefs. That the players systematically form qualitatively different biases is mysterious, though the number of models that can rationalize this obser-
vation is potentially large. What is the true mechanism?

We argue that the roles subjects find themselves in causally induce different degrees of strategic sophistication in the level \( k \) sense (e.g. Nagel [1995] and Stahl and Wilson [1995]). In particular, player 1-subjects are made more sophisticated than player 2-subjects, and this generates precisely the biases we observe: whereas player 2-subjects overwhelmingly attribute the low-level action to player 1 (\( U \) when \( X \) is large, \( D \) when \( X \) is small), a sizable fraction of player 1-subjects correctly anticipate this.

We provide two types of corroborating evidence for this sophistication hypothesis. First, all subjects stated beliefs in both roles of a dominance solvable game, from which we derive a subject-level measure of strategic sophistication that is collected identically for all subjects. By this measure, player 1-subjects are much more sophisticated than player 2-subjects. Second, player 1-subjects have much longer response times, suggestive of deeper thinking.

We first present this evidence. Then, we argue that player 1-subjects’ stated beliefs in the \( X \)-games also suggest much higher levels of strategic sophistication. To this end, we fit a structural model to each subject’s beliefs data from the \( X \)-games and show, in the context of the model, that this implies much higher levels of sophistication for player 1-subjects. We then validate this finding by showing that the implied measures of sophistication correlate strongly with those measured in the dominance solvable games.\(^{32}\)

The data tell us that player role itself has a causal effect on sophistication but it cannot tell us the precise mechanism. We conclude this section by discussing potential mechanisms and suggestions for future work.

### 6.2.1 Dominance solvable games and a measure of sophistication

If player 1-subjects are truly made more sophisticated because of their role in the \( X \)-games, we conjecture that this should “spill over” to other games.\(^{33}\) To this end, we consider the dominance solvable games reproduced in Figure 12.

Games \( D_1 \) and \( D_2 \) (“Dominant 1” and “Dominant 2”) are identical up to which player faces which set of payoffs. In the former, player 1 has a strictly dominant action and in the latter, player 2 has a strictly dominant action. Furthermore, in game \( D_i \), one of player \( j \)’s actions is the best response to a uniform distribution and the other is the best response to

\(^{32}\)This is internally consistent in that the structural model predicts such a correlation between the measures.

\(^{33}\)One concern is that, since experience in the fully mixed \( X \)-games affects behavior in the dominance solvable \( D_1 \) and \( D_2 \), these latter games may also have an affect on behavior in the former. However, we find this implausible since the \( X \)-games take up a large majority of the experiment, so we think of behavior in \( D_1 \) and \( D_2 \) as reflections of the cognitive processes used in the \( X \)-games.
Figure 12: Dominance solvable games. In game $Di$, player $i$ has a strictly dominant action (taken by levels $k \geq 1$). Player $j$ can either best respond to a uniform distribution ($k = 1$) or to player $i$’s dominant action ($k \geq 2$).

In the level $k$ framework of strategic sophistication (e.g. Nagel [1995] and Stahl and Wilson [1995]), level 0 is assumed to uniformly randomize, level 1 best responds to level 0, and so on, with level $k$ best responding to level $k - 1$. In game $Di$, the following characterizes level-types. Player $i$: levels $k \geq 1$ take the dominant action. Player $j$: level 1 best responds to a uniform distribution and levels $k \geq 2$ best respond to $i$’s dominant action.

This suggests two benchmark beliefs: (1) $i$’s belief that $j$ takes her dominant action in $Dj$ and (2) $i$’s belief that $j$ best responds to $i$’s dominant action in $Di$. Assuming $i$ believes $j$ is drawn from a distribution of level types, for any fixed probability that $i$ believes $j$ is level 0, these correspond to increasing functions of $i$’s belief that $j$ is any level $k \geq 1$ and $i$’s belief that $j$ is any level $k \geq 2$, respectively. We call these benchmark beliefs $\beta(k \geq 1)$ and $\beta(k \geq 2)$, and they are readily seen as coarse measures of sophistication as they measure the belief that the opponent is of a sufficiently high level.

Throughout the paper, we aggregate $\hat{\beta}(k \geq 1)$ and $\hat{\beta}(k \geq 2)$ to the subject level by averaging beliefs across instances of $Di$ and $Dj$, respectively (player $i$ sees $Di$ three times and $Dj$ two times in the second stage). Figure 13 gives histograms of these measures for each player role.

From the top panel of Figure 13, we see that both players have very similar distributions of $\hat{\beta}(k \geq 1)$ that are highly concentrated toward the right of the space with modes close to 100 and very similar means of approximately 85 (solid lines). The corresponding action frequencies (from $[A, o]$) are even higher: greater than 95 for both players (dashed lines). From the bottom panel of Figure 13, we see that player 1’s distribution of $\hat{\beta}(k \geq 2)$ is relatively uniform whereas that of player 2 is concentrated below 50, and the respective means are 56 and 33 (solid lines). The corresponding action frequencies are nearly the same for both players at approximately 78 (dashed lines).

Our main takeaways from Figure 13 are twofold. First, there is much more variation in
Figure 13: Sophistication by player. The top panel gives histograms of $\hat{\beta}(k \geq 1)$, i’s belief that $j$ best responds to his dominant action in $Dj$, across subjects. The bottom panel gives histograms of $\hat{\beta}(k \geq 2)$, i’s belief that $j$ best responds to i’s dominant action in $Di$ (as opposed to the a uniform distribution), across subjects. The solid lines mark i’s average beliefs, and the dashed lines mark $j$’s corresponding action frequencies from $[A, o]$. $\hat{\beta}(k \geq 2)$ than in $\hat{\beta}(k \geq 1)$. In other words, subjects overwhelmingly believe other subjects respond to incentives, but vary greatly in how many additional steps of reasoning they perform. Therefore, we will use $\hat{\beta}(k \geq 2)$ as our measure of sophistication.

Second, player 1 is much more sophisticated than player 2 by this measure, with an average difference in $\hat{\beta}(k \geq 2)$ of 22. In Table 19 of Appendix 11.7, we show that this sophistication gap is highly significant, robust to various controls, and not driven by erratic subjects.

Importantly, since $D1$ and $D2$ are exactly the same up to which player faces which payoffs, the sophistication measure is derived in exactly the same way for both players. Furthermore,
all subjects observed exactly the same games throughout the experiment and were randomly assigned to their roles. Thus, the difference in measured sophistication must be caused by their experience in different roles of the X-games.

The frequency of actions taken in the dominance solvable games are nearly identical across player 1- and player 2-subjects in the first stage. That stated beliefs (and to some extent actions) differ across player roles in the second stage suggests that role-dependent no-feedback learning took place. Interestingly, however, there is no evidence of learning throughout the experiment in the sense that, within each stage of the experiment, there is no within-player trend in actions or beliefs across multiple rounds of the same game. Hence, we believe that there was some no-feedback “belief learning” in the first stage that did not manifest in actions. In the second stage, player 1-subjects’ stated beliefs already indicated higher levels of sophistication in the very first instance of Di, so we believe all of the learning had already taken place by that point.

6.2.2 Response times

For additional support, we consider response times. If the player 1-role induces greater strategic sophistication, we would expect for player 1 to also take longer to form beliefs since they go farther in terms of strategic reasoning.

In Appendix Figure 30, we plot the average time to finalize belief statements for each game and player role. Player 1 takes longer on average for all games. That player 1-subjects take longer on games Di and Dj is further suggestion that their experience in the role of player 1 in the X-games spills over to these new environments even though they are the same for both players.

6.2.3 The relationship between sophistication and behavior

If differential sophistication across player roles is to explain behavior in the X-games, sophistication measured in dominance solvable games should be predictive of behavior in the X-games within-role. We establish this before formally modeling the relationship between sophistication and beliefs in the next subsection.

To this end, we divide player 1-subjects into low and high sophistication groups based on having values of \( \hat{\beta}(k \geq 2) \) below and above the player 1-median, and similarly for player 2-subjects. We then compare behaviors across the sophistication groups, focusing on beliefs data and first-stage actions data \([A, BA]\) (results using the second-stage actions are similar). Appendix Table 20 summarizes our results, which we discuss in detail below. In each column,
we regress beliefs or actions on indicators for each of the six $X$-games (omitted from the table) and indicators for each of the six games interacted with an indicator for the high sophistication group. The results are robust to alternate groupings and using $\hat{\beta}(k \geq 2)$ as a continuous variable.

Compared to less sophisticated player 1s, more sophisticated player 1s tend to believe that player 2 plays $L$ less often for $X > 20$ (more often for $X < 20$). Compared to less sophisticated player 2s, more sophisticated player 2s tend to believe that player 1 plays $U$ less often for $X > 20$ (more often for $X < 20$). In Appendix Figure 31, we plot histograms of beliefs by sophistication group for both players and all games. This shows that sophisticated player 1s have more extreme beliefs while for player 2, it is the opposite. Hence, low sophistication does not simply proxy for more conservative beliefs.

In Figure 14, we plot the empirical action frequencies from $[A,BA]$. For player 1, the difference between high and low sophistication groups is quantitatively very large, highly significant (column 2 of Appendix Table 20), and qualitatively surprising. Consistent with the differences in their beliefs, the low sophistication group tends to take $U$ for $X > 20$ and $D$ for $X < 20$, while for the high sophistication group, it is the opposite. Interestingly, the low sophistication group is very consistent with the joint QRE-NBE predictions and the high sophistication group is not simply proxy for more conservative beliefs.

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$^{34}$We tried terciles and quartiles as well as using the median of $\hat{\beta}(k \geq 2)$ across all subjects for both players instead of player-specific medians.
sophistication group looks bizarre from the perspective of this theory. On average, however, the behavior is not far from Nash (left panel of Figure 3). For player 2, the two sophistication groups have very similar actions data which cannot be distinguished statistically (column 4 of Appendix Table 20).

6.2.4 Beliefs and sophistication in the X-games

Our analysis shows that player 1-subjects are more sophisticated than player 2-subjects in the dominance solvable games and that beliefs in the dominance solvable games predict behavior in the X-games. However, this does not imply in of itself that player 1-subjects form more sophisticated beliefs in the X-games. To determine if this is the case, we introduce a simple model of belief formation that provides a formal link between beliefs in the X-games and sophistication. The goal is not to propose a general theory of belief formation, but to infer sophistication from the X-game data to test the hypothesis that player 1 is more sophisticated than player 2 and determine if this can generate the biases we observe. As such, the model is highly specialized to the X-games.

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</table>

Table 10: Levels in game $X$.

To this end, we will use a modified cognitive hierarchy framework (Camerer et al. [2004b]) in which each subject believes she faces opponents drawn from a distribution of level types (e.g. Stahl and Wilson [1995] and Nagel [1995]). Recall that level 0 is uniformly randomizes, level 1 best responds to level 0, and so on, with level $k$ best responding to level $k - 1$. In forming beliefs, subjects believe they face a distribution of level types. We say that a subject is sophisticated if her beliefs imply that she believes she faces types with high levels.

In Table 10, we write out the actions taken by different level $k$-types in the X-games, written as $\sigma^U_k$ and $\sigma^L_k$. In the case of indifference we assume uniform tie-breaking. We make two observations. First, the levels “cycle” in the sense that $\sigma^U_0 = \sigma^U_1$ for even $k$ and $\sigma^L_0 = \sigma^L_1$ for odd $k$. Second, $X$ matters only insofar as it is greater than or less than 20.

The fact that levels cycle causes problems for identification: any action can be interpreted as coming from an arbitrarily high level and therefore any belief can be rationalized as being
arbitrarily sophisticated. We restore identification by truncation.

Camerer et al. [2004b] find that an average of 1.5 steps of reasoning fits the data from many games, so we assume player \( i \) forms beliefs according to the following three step procedure. First, player \( i \) imagines what \( j \) would do naively, which we assume is a best response to uniform play. Second, player \( i \) imagines her own best response to that action. Third, player \( i \) imagines \( j \)'s best response to that action. During this process, player \( i \) imagines player \( j \) taking the level 1 and level 3 actions, so we assume player \( i \) believes she faces a fraction \((1 - \alpha)\) of level 1s and a fraction \( \alpha \) of level 3s. Player \( i \)'s belief in game \( X \) is thus given by

\[
\bar{\sigma}^X_j(\alpha) = (1 - \alpha) \cdot \sigma^1_j X + \alpha \cdot \sigma^3_j X ,
\]

where \( \alpha \) is a free parameter that we interpret as sophistication as it is the belief that the opponent is of a high level. Our decision to begin the process of introspection at level 1, as opposed to level 0, is motivated by the fact that subjects overwhelmingly expect their opponents to take the dominant action in \( D_j \) (top panel of Figure 13), which is unsurprising in simple games. That the introspection process skips level 2 is a consequence of iterated best response in asymmetric games, not an arbitrary restriction.

The fact that \( \sigma^{k,X}_j \) only depends on \( X \) insofar as \( X \) is greater or less than 20 means that beliefs formed as in (6) will also have this property. However, this is counterfactual: the analysis of Section 5.3 shows that beliefs change systematically across all values of \( X \). For this reason, we generalize level \( k \) to allow for each level type to make payoff sensitive errors.\(^{35}\) Player \( i \) believes player \( j \) of level \( k \) faces a vector of payoffs \( v^k_j = (v^k_{j1}, v^k_{j2}) \) and takes an action according to

\[
Q^\mu_j(v^k_j) = \frac{(v^k_{j1})^{1/\mu}}{(v^k_{j1})^{1/\mu} + (v^k_{j2})^{1/\mu}} ,
\]

where parameter \( \mu > 0 \) controls the sensitivity to payoff differences. The action of player \( j \) of level \( k \) is thus defined recursively according to \( \sigma^{k,X}_j(\mu) \equiv Q^\mu_j(\bar{u}_j(\sigma^{k-1}_i(\mu; X))) \) and \( \sigma^0_j = \frac{1}{2} \). We replace \( \sigma^{k,X}_j \) in (6) with \( \sigma^{k,X}_j(\mu) \) which yields

\[
\bar{\sigma}^X_j(\alpha, \mu) = (1 - \alpha) \cdot \sigma^{1,X}_j(\mu) + \alpha \cdot \sigma^{3,X}_j(\mu).
\]

We still interpret \( \alpha \) as sophistication and \( \mu \) is the payoff sensitivity player \( i \) attributes to \( j \). We favor the Luce form of quantal response (7) because it is scale invariant, which implies

\(^{35}\)The idea of injecting noise into the description of levels is not new. See, for example, Capra [1999], Weizsacker [2003], and Rogers et al. [2009].
beliefs described by (8) are symmetric in $\sigma_{l}^{NE}$—a feature we will show matches the data.

To gain some intuition for the types of beliefs implied by (8), we first consider the cases of $\mu = 0$ and $\mu = 1$. When $\mu = 0$, player $i$ believes each level-type of player $j$ best responds to their beliefs: (8) collapses to (6) and hence $X$ only matters insofar as $X$ is greater than or less than 20 or equivalently if $\sigma_{l}^{NE}$ is less than or greater than $\frac{1}{2}$. Beliefs thus follow a step pattern:

$$\tilde{\sigma}_{U}^{X}(\alpha, 0) = \begin{cases} 
(1 - \alpha) & \sigma_{l}^{NE, X} < \frac{1}{2} \\
\alpha & \sigma_{l}^{NE, X} > \frac{1}{2} 
\end{cases}$$

$$\tilde{\sigma}_{L}^{X}(\alpha, 0) = \begin{cases} 
\frac{1}{2}(1 - \alpha) & \sigma_{l}^{NE, X} < \frac{1}{2} \\
\frac{1}{2}(1 + \alpha) & \sigma_{l}^{NE, X} > \frac{1}{2} 
\end{cases}.$$  

For $\mu > 0$, player $i$ believes each level-type of player $j$ makes payoff sensitive errors in best responding to her beliefs. Hence, $i$’s beliefs are sensitive to all changes in $X$ and therefore also to changes in $\sigma_{l}^{NE}$. When $\mu = 1$, properties of (7) imply that $i$’s beliefs are linear in $\sigma_{l}^{NE}$:

$$\tilde{\sigma}_{U}^{X}(\alpha, 1) = (1 - \alpha) \cdot (1 - \sigma_{l}^{NE, X}) + \alpha \cdot \frac{1}{2}$$

$$\tilde{\sigma}_{L}^{X}(\alpha, 1) = (1 - \alpha) \cdot \frac{1}{2} + \alpha \cdot \sigma_{l}^{NE, X}.$$  

Thus, when beliefs are viewed as functions of $\sigma_{l}^{NE}$, identification of sophistication is based on levels for $\mu = 0$ and based on the slope for $\mu = 1$. Note also that, in the $\mu = 1$ case, beliefs coincide with Nash equilibrium when players are fully sophisticated ($\alpha = 1$). In general, it is easy to generate parameter values for which the beliefs fall in the interior of the QRE-NBE region.

### 6.2.5 Structural model

We adapt the model of the previous section to be fit to individual subjects’ belief data from the X-games. By fitting the model to each subject’s data, we infer a measure of strategic sophistication for each subject.

We recast the belief $\tilde{\sigma}_{j}^{X}$ defined in (8) as the central tendency of beliefs and assume that beliefs are noisy with a parametric error structure. For player $i$ with parameters $\alpha$ and $\mu$, 

48
we assume belief \( b \in \{0, 1, \ldots, 100\} \) is drawn in game \( X \) according to
\[
p_X(b; \alpha, \mu, \lambda) = \frac{e^{-\lambda(b-100-\bar{\sigma}_j^X(\alpha, \mu))^2}}{\sum_{b' \in \{0,1,\ldots,100\}} e^{-\lambda(b'-100-\bar{\sigma}_j^X(\alpha, \mu))^2}},
\]
so that the belief closest to \( 100 \cdot \bar{\sigma}_j \) is the mode and \( \lambda > 0 \) is a precision parameter.

The data of subject \( s \) in role \( i \) is a set of 30 belief statements \( \{\hat{b}_{si}^X\}_{lX} \) where \( l \in \{1, \ldots, 5\} \) indexes each elicitation and \( X \) indexes the game. For each subject, we choose \( \alpha, \mu, \) and \( \lambda \) to maximize the log-likelihood of stated beliefs:
\[
L^s(\hat{b}; \alpha, \mu, \lambda) = \sum_{X} \sum_{l=1}^{5} \ln(p_X(\hat{b}_{si}^X; \alpha, \mu, \lambda)).
\]
We find that for 50 out of 110 subjects (45%), a likelihood ratio test rejects the restriction \( \mu = 0 \) at the 5% level, meaning that a substantial fraction of subjects are significantly sensitive to variations in \( X \) in ways allowed by the model.

Figure 15 plots the beliefs of some representative subjects, superimposed with the best-fit \( \bar{\sigma}_j^X(\hat{\alpha}, \hat{\mu}) \) for \( X \in (0, \infty) \). The top panels are for player 2-subjects forming beliefs about player 1, and the bottom panels are for player 1-subjects forming beliefs about player 2. Clearly, there is considerable heterogeneity across subjects, but the model is flexible enough to accommodate their diverse belief patterns.

The top panel of Figure 15 features player 2-subjects who, from left to right, are increasing in inferred sophistication \( \hat{\alpha} \). Subject 105 believes, overwhelmingly, that player 1 will take \( U \) when \( X \) is large and take \( D \) when \( X \) is small, or in other words to engage in level 1 behavior. For this reason, the model infers the low level of sophistication \( \hat{\alpha} = 0.04 \). At the other extreme, subject 104 believes that player 1 will mostly take \( D \) when \( X \) is large and take \( U \) when \( X \) is small, or in other words to engage in level 3 behavior, and so the model infers a high level of sophistication \( \hat{\alpha} = 0.76 \). Interestingly, it is the very sophisticated subjects whose beliefs systematically fall outside of the QRE-NBE region.

The bottom panel of Figure 15 features player 1-subjects who, from left to right, are increasing in inferred sophistication \( \hat{\alpha} \). Subject 26 believes that player 2 will tend to take \( R \) when \( X \) is large and take \( L \) when \( X \) is small, consistent with a mix of level 1 and level 3 behavior. The same can be said of Subject 5, however, the model infers that Subject 5 is much more sophisticated than Subject 26, with values of \( \hat{\alpha} = 1 \) and \( \hat{\alpha} = 0.47 \), respectively. Inspecting their beliefs more closely, we observe that Subjects 26 and 5 have estimated
Figure 15: Individual subjects’ beliefs. We plot representative individual subjects’ stated beliefs, superimposed with the best fit $\hat{\sigma}_j^X(\hat{\alpha}, \hat{\mu})$. The top row is for player 2-subjects forming beliefs about player 1, and the bottom row is for player 1-subjects forming beliefs about player 2.
sensitivities of $\hat{\mu} \approx 0$ and $\hat{\mu} \approx 1$, respectively, corresponding to step-like and linear belief patterns. Hence, while the two subjects may have similar beliefs when averaged across games $X > 20$ (and similar beliefs when averaged across games $X < 20$), Subject 5 believes in a much higher fraction of level 3 opponents, albeit much noisier ones.

6.2.6 Player 1 is more sophisticated than player 2 in the $X$-games

We show that the structural model applied to the $X$-games implies that player 1-subjects tend to be much more sophisticated by $\hat{\alpha}$ than player 2-subjects. This can be seen from Figure 16, which replicates the bottom panel of Figure 13 by plotting histograms of inferred sophistication $\hat{\alpha}$ for player 1- and player 2-subjects.

![Figure 16: Inferred sophistication by player.](image)

To further validate this finding, we show that $\hat{\alpha}$ estimated from the fully mixed $X$-games and $\hat{\beta}(k \geq 2)$ directly measured in dominance solvable $Di$ are strongly correlated. Applying the structural model to $Di$, $\alpha$ and $\mu$ cannot be separately identified in the sense that $\beta(k \geq 2)$ may be consistent with different $(\alpha, \mu)$-pairs.$^{36}$ However, for any fixed $\mu$, $\beta(k \geq 2)$ is an increasing function of $\alpha$. Hence, if $\mu$ is sufficiently uncorrelated with $\alpha$, $\hat{\beta}(k \geq 2)$ is predicted to correlate with $\hat{\alpha}$. We are therefore justified in comparing $\hat{\alpha}$ and $\hat{\beta}(k \geq 2)$ to validate the structural model.

Figure 17 gives a scatter plot of inferred versus directly measured sophistication. We find there is a strong positive correlation for player 1-subjects, player 2-subjects, and all subjects; and this is confirmed in Table 11 which presents the correlations.

---

$^{36}$This is not an issue in the $X$-games because of variations in $X$.  

51
Figure 17: Inferred versus directly measured sophistication. We give a scatterplot of subjects’ \( \hat{\alpha} \) versus \( \hat{\beta}(k \geq 2) \), with best-fit lines for player 1-subjects, player 2-subjects, and all subjects.

<table>
<thead>
<tr>
<th>( \alpha, \mu )</th>
<th>Player 1 Pearson</th>
<th>Player 1 spearman</th>
<th>Player 2 Pearson</th>
<th>Player 2 spearman</th>
<th>Both Pearson</th>
<th>Both spearman</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha, \mu )</td>
<td>0.53***</td>
<td>0.54***</td>
<td>0.44***</td>
<td>0.34***</td>
<td>0.57***</td>
<td>0.54***</td>
</tr>
<tr>
<td>( \alpha, \mu = 0 )</td>
<td>0.48***</td>
<td>0.50***</td>
<td>0.41***</td>
<td>0.35***</td>
<td>0.50***</td>
<td>0.46***</td>
</tr>
<tr>
<td>( \alpha, \mu = 1 )</td>
<td>0.52***</td>
<td>0.54***</td>
<td>0.41***</td>
<td>0.34***</td>
<td>0.57***</td>
<td>0.55***</td>
</tr>
</tbody>
</table>

Table 11: Correlation between inferred and directly measured sophistication

For robustness, in addition to the 2-parameter \((\alpha, \mu)\) model, we also consider the 1-parameter restrictions \((\alpha, \mu = 0)\) and \((\alpha, \mu = 1)\) discussed in the previous section. We also report both Pearson (linear) coefficients and spearman (rank-based) coefficients. We find that the correlations are in all cases highly significant, with large magnitudes ranging from 0.34 to 0.63. The correlations are a bit higher for player 1 than for player 2, and highest for both players pooled together. The 2-parameter model and restricted \(\mu = 1\) model lead to very similar correlations, which are slightly higher than the correlations implied by the \(\mu = 0\) restricted model.

Taken together, the results of this section provide support for the sophistication hypothesis. The structural model is qualitatively consistent with the patterns observed in subjects’ beliefs data, it captures the stylized fact of the sophistication gap, and it implies degrees of sophistication that correlate strongly with the direct measures.
6.2.7 Discussion

Our results indicate that player 1 forms more sophisticated beliefs than player 2 in the X-games. Since the subjects in the two roles were ex-ante identical, this suggests a model of *endogenous* role-dependent sophistication. It is beyond the scope of this paper, but we consider developing such models collecting datasets to differentiate between them is a promising direction for future research.

It seems difficult to reconcile our data with a rational, optimizing model of endogenous sophistication (e.g. the model of Alaoui and Penta [2015]) for the reason that one player or the other faces much higher average payoffs depending on X, and yet it is always player 1 who forms more sophisticated beliefs. We believe psychological explanations related to the salience of player 1’s payoffs are more promising.

6.3 Modeling actions and beliefs jointly

In Sections 6.1 and 6.2, we offered explanations for the failures of *monotonicity* and *unbiasedness*, respectively. These explanations came with structural models that were fit to actions given beliefs and then to beliefs, respectively. In this section, we combine the previously introduced elements to maximize the likelihood of actions and beliefs jointly, which we show can rationalize the whole of the data, including the belief biases we observe.

The data of subject s in role i is a set of 30 action-belief pairs \( \{\hat{\alpha}_{si}^X, \hat{\beta}_{si}^X\}_{lX} \) where \( l \in \{1, \ldots, 5\} \) indexes each elicitation and X indexes the game. For each player 1-subject s, we choose \( \rho, \mu_a, \alpha, \mu, \lambda \) to maximize

\[
L^s(\hat{\alpha} | \hat{\beta}; \cdot) = \sum_X \sum_{l=1}^5 \ln(p_X(\hat{\alpha}_{si}^X | \hat{\beta}_{si}^X; \rho, \mu_a)p_X(\hat{\beta}_{si}^X; \alpha, \mu, \lambda))
\]

\[
= \sum_X \sum_{l=1}^5 \ln(p_X(\hat{\alpha}_{si}^X | \hat{\beta}_{si}^X; \rho, \mu_a)) + \sum_X \sum_{l=1}^5 \ln(p_X(\hat{\beta}_{si}^X; \alpha, \mu, \lambda))
\]

\[
= L^s(\hat{\alpha} | \hat{\beta}; \rho, \mu_a) + L^s(\hat{\beta}; \alpha, \mu, \lambda),
\]

where \( L^s(\hat{\alpha} | \hat{\beta}; \cdot) \) and \( L^s(\hat{\beta}; \cdot) \) are as before. Hence, we find the same parameter estimates as before for each subject. For player 2-subjects, we fit the same model, except under the assumption of linear utility \( \rho = 0 \) as curvature cannot be identified due to the symmetry of player 2’s payoffs (see Section 6.1).
After fitting the model to each subject, we simulate the aggregate data. In Figure 18, we plot the simulated empirical frequencies and median beliefs, which we compare to the data from [A,BA] to which the model was fit. We find that the model generates the observed belief biases, and we already know from Section 6.2.6 that the fitted model implies much higher levels of sophistication for player 1-subjects.

7 Action-noise or belief-noise?

Clearly, there is considerable noise in both actions and beliefs. What is lost in ignoring either source of noise?

In this section, we explore this question via a counterfactual exercise. Specifically, we construct two counterfactual action frequencies that result from “turning off” just one source of noise in the second-stage data [A,BA] (for which we can associate actions with beliefs). $\tilde{\sigma}_{best\ response}^i$ is what we would observe if subjects best responded to every stated belief, and $\tilde{\sigma}_{correct\ beliefs}^i$ is what we would observe if subjects had correct beliefs (over first-stage actions [A, o]). The former can be constructed directly from the data. To construct the latter, we set $i$’s beliefs equal to $j$’s empirical action frequency and assume actions are governed by the random utility model with curvature fitted to each subject’s data from Section 6.1.

In Figure 19, we plot $\tilde{\sigma}_{[A,BA]}^i$, $\tilde{\sigma}_{best\ response}^i$, and $\tilde{\sigma}_{correct\ beliefs}^i$ for both players. As a measure
of the performance of each counterfactual, we consider the average absolute differences between actual and counterfactual frequencies across the games for each player, $D_{i}^{\text{best response}} \equiv \frac{1}{6} \sum_{X} |\hat{\sigma}_{i}^{[A,B]}|_{X} - \hat{\sigma}_{i}^{\text{best response},X}$ and $D_{i}^{\text{correct beliefs}} \equiv \frac{1}{6} \sum_{X} |\hat{\sigma}_{i}^{[A,B]}|_{X} - \hat{\sigma}_{i}^{\text{correct beliefs},X}$. These represent the prediction errors or loss in ignoring action-noise and belief-noise, respectively, and are displayed in Table 12.

![Figure 19: Ignoring action-noise and belief-noise—a counterfactual.](image)

We see that, for both players, the two counterfactuals are fairly inaccurate, indicating that action-noise and belief-noise are both important ingredients for explaining behavior. Models that ignore any one—as indeed the large majority of models applied to experimental data do—may suffer from misspecification. Interestingly, which source of noise is more important depends on player role. For player 1, $\hat{\sigma}_{i}^{\text{correct beliefs}}$ is much more accurate than $\hat{\sigma}_{i}^{\text{best response}}$, whereas for player 2, it is the opposite. This is intuitive as player 1’s stated beliefs are fairly accurate but she faces a difficult decision for any given belief, and it is the opposite for player 2.

<table>
<thead>
<tr>
<th>Player</th>
<th>$D_{i}^{\text{best response}}$</th>
<th>$D_{i}^{\text{correct beliefs}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>player 1 ($i = 1$)</td>
<td>0.30</td>
<td>0.16</td>
</tr>
<tr>
<td>player 2 ($i = 2$)</td>
<td>0.13</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Table 12: Loss in ignoring action-noise and belief-noise.
8 Issues of belief elicitation: a discussion

Our analysis depends on making beliefs observable through direct elicitation, so we discuss two well-known and potentially confounding issues of belief elicitation. First, it may be that stated beliefs are only noisy signals of subjects’ underlying latent or “true” beliefs. Second, belief elicitation itself may affect the actions subjects take. We argue that these issues do not affect our main conclusions.

8.1 Stated beliefs as noisy signals of true beliefs

Throughout the paper, we have implicitly assumed that stated beliefs equal the latent or “true” beliefs that subjects hold in their minds and guide their actions. More generally, it may be that stated beliefs are noisy signals of the underlying true beliefs due to errors in reporting or noisy introspection about one’s beliefs (see Rutstrom and Wilcox [2009] for a discussion). In that case, can we still say that the unobserved true beliefs are noisy? Can we reject the same axioms with respect to the true beliefs? We argue that the answer to both questions is yes.

We suppose that, for a given game, $b^*_s$ and $b^*_0$ are stated and true beliefs, respectively. These are (possibly degenerate) random variables whose support is contained in $[0,1]$. Let $b_0$ be a realization of true beliefs, and let $b^*_s(b_0)$ be the random stated beliefs conditional on $b_0$. We assume that actions depend on true belief realizations through the function $Q_i(\bar{u}_i(b_0))$.

Are true beliefs noisy? If within-subject-game, the true belief were fixed and stated beliefs were simply noisy signals of the underlying belief, then within-subject-game variation in stated beliefs would not be predictive of actions. If this were the case, we would see coefficients of 0 in Table 7, but this is strongly rejected. Hence, we conclude that true beliefs are noisy.

As we found using stated beliefs, are monotonicity and unbiasedness also rejected with respect to true beliefs? To answer this, we require additional structure. To this end, assume that stated beliefs are drawn from a distribution that is centered, in the sense of median, around the true belief realization: $\text{med}[b^*_s(b_0)] = b_0$ for all $b_0$ (and $b^*_s(b_0) = b_0$ w.p. 1 if $b_0 \in \{0, 100\}$). Under this assumption, we argue that it is very unlikely that either axiom holds in true beliefs given our data.

Consider player 1 in game X5 (see Figure 6). The indifferent belief is 80, and the monotonicity violation occurs in the interval of stated beliefs [60, 80]. Suppose that actions given true beliefs are governed by $Q_i(\bar{u}_i(b_0)) = 1/2$ for $b_0 \leq 80$ and $Q_i(\bar{u}_i(b_0)) = 1$ for $b_0 >$
80, which is the monotonic quantal response function most likely to generate the observed violation. Under the assumption that \( \text{med}(\hat{b}^*_s(\hat{b}_0)) = b_0 \) for all \( b_0 \), the expected mass of stated beliefs in [60, 80] that is associated with a true belief \( b_0 > 80 \), and thus the action \( Q_i = 1 \), is at most equal to the mass of stated beliefs greater than 80. It is clear that this is insufficient to rationalize the violation we observe. Hence, the underlying \( Q_i \) defined over true beliefs cannot be monotonic.

We found that player 2 forms very biased stated beliefs over player 1’s actions. For instance, in X80, \( \text{med}(\hat{b}^*_s) > \hat{\sigma}_U \) (see top left panel of Figure 9). Suppose that, in true beliefs, \( \text{med}(b^*_s) = \hat{\sigma}_U \). This does not imply that \( \text{med}(\hat{b}^*_s) = \hat{\sigma}_U \), but it does imply that \( \mathbb{P}(\hat{b}^*_s > \hat{\sigma}_U) \leq \frac{3}{4} \).\(^{37}\) and we observe that \( \hat{\mathbb{P}}(\hat{b}^*_s > \hat{\sigma}_U) \) is much greater than three-fourths in the data. Hence, the underlying true beliefs cannot be unbiased.

8.2 The effects of belief elicitation

There is little consensus on if, how, and under what conditions belief elicitation has an effect on the actions subjects take. In their recent review articles on belief elicitation, Schlag et al. [2015] describes the evidence as “scanty and contradictory” whereas Schotter and Trevino [2014] state that the “evidence presents a more consistent picture in favor of the idea that belief elicitation is innocuous”. We are unaware of studies that elicit beliefs for asymmetric matching pennies without the influence of feedback, and, for the studies that have used feedback, the documented effects have been small.\(^{38}\)

It has been conjectured that belief elicitation may increase strategic sophistication (see the discussion in Schlag et al. [2015]), but to the best of our knowledge, no previously documented effects can readily be interpreted in this way.\(^{39}\)

In Appendix 11.4, we show that there are small, but systematic and statistically significant, differences between the first- and second-stage action frequencies in [A,A]. We find no such differences between the two stages of the [A,A] treatment that did not involve belief

\(^{37}\)That \( \text{med}(b^*_0) = \hat{\sigma}_U \) implies that \( \mathbb{P}(\hat{b}^*_0 > \hat{\sigma}_U) = \mathbb{P}(\hat{b}^*_0 < \hat{\sigma}_U) = \frac{1}{2} \). Given that \( \text{med}(b^*_0(\hat{b}_0)) = b_0 \) for all \( b_0 \), \( \mathbb{P}(\hat{b}^*_0 > \hat{\sigma}_U) \) is maximized when \( \mathbb{P}(\hat{b}^*_0(\hat{b}_0) > \hat{\sigma}_U|\hat{b}_0 > \hat{\sigma}_U) = 1 \) and \( \mathbb{P}(\hat{b}^*_0(\hat{b}_0) > \hat{\sigma}_U|\hat{b}_0 < \hat{\sigma}_U) = \frac{1}{2} \), which implies that \( \mathbb{P}(\hat{b}^*_0 > \hat{\sigma}_U) = \frac{3}{4} \).

\(^{38}\)Nyarko and Schotter [2002] find no effect. Rutstrom and Wilcox [2009] find an effect for only one player and only during early rounds. They claim to be the first to find any such effect in games with unique, mixed strategy Nash equilibria, and we are unaware of any studies to do so since.

\(^{39}\)Costa-Gomes and Weizsacker [2008] find, in the context of 3 x 3 dominance solvable games, that beliefs seem more sophisticated than the corresponding actions, but they also find that the belief elicitation has no effect on actions. Schotter and Trevino [2014] suggest that belief elicitation may hasten convergence to equilibrium in games played with feedback, but this is distinct from sophistication.
elicitation, and so we conclude that there is a belief elicitation effect. In Appendix 11.4, we provide additional discussion and argue that this does not affect our main conclusions.

9 Relationship to the existing literature

A central goal of behavioral game theory (Camerer [2003]) is to describe how real people play games. This paper contributes to the large sub-literature that focuses on bounded rationality as drivers of behavior (as opposed to, for example, other-regarding preferences (Fehr and Schmidt [1999])).

We fit most squarely in the literature on stochastic equilibrium models that maintain fixed-point consistency between players’ actions but allow for random elements. The prominent example is quantal response equilibrium (QRE), a concept that allows for “noise in actions” but maintains that beliefs are correct. Early QRE theory was developed in a series of papers (McKelvey and Palfrey [1995], McKelvey and Palfrey [1998], Chen et al. [1997], and others) and is surveyed in a recent monograph (Goeree et al. [2016]).

Many papers acknowledge that the assumption of correct beliefs is unrealistic, but the large majority of these papers applied to experimental data are non-equilibrium models such as level k (e.g. Nagel [1995] and Stahl and Wilson [1995]; and reviewed in Crawford et al. [2013]) and its many successors (e.g. Camerer et al. [2004b], Alaoui and Penta [2015], and Goeree and Holt [2004]). These models have proven extremely useful in explaining experimental data post-hoc, but their application is sometimes criticized for lacking the discipline that equilibrium consistency brings.

There are many equilibrium models that involve biased or otherwise incorrect beliefs (e.g. Geanakoplos et al. [1989] and Heller and Winter [2018]), but these are typically ill-suited for (nor were they designed for) direct application to experimental data. In terms of models that allow for “noise in beliefs”, there are very few. An early example is the parametric sampling equilibrium (Rubinstein and Osborne [2003]) which has been applied to experimental data (Selten and Chmura [2008]). Notably, Friedman and Mezzetti [2005] introduce the notion of a belief-map as part of their random belief equilibrium (RBE). Their focus is on the limiting case in which belief-noise “goes to zero” to develop a theory of equilibrium selection, so their conditions on belief distributions do not impose any testable restrictions beyond ruling out weakly dominated actions. Noisy belief equilibrium (NBE) (Friedman [2019]), on which our paper builds, was developed to study the case in which belief noise is bounded away from zero, so it maintains the structure of the belief-map but imposes behavioral axioms to impose
testable restrictions.

We are novel in studying an equilibrium model that allows for both noise in actions and noise in beliefs. Because it is a non-parametric, axiomatic model that makes set-predictions and we study the restrictions imposed by the axioms, there is a clear relationship to the literature on the empirical content of QRE (e.g. Haile et al. [2008], Goeree et al. [2005], Melo et al. [2017], Goeree and Louis [2018], and Goeree et al. [2018]).

We also make several contributions to the literatures on belief elicitation and strategic sophistication. By having multiple belief elicitations per subject-game without feedback, we are able to study noise in beliefs. By eliciting beliefs for a family of closely related games, we are able to track how beliefs vary within-subject across games and compare these belief patterns across individual subjects. This distinguishes us from experiments that elicit beliefs once for each game in a set of seemingly unrelated games (e.g. Costa-Gomes and Weizsacker [2008] and Rey-Biel [2009]) as well as studies that elicit beliefs for the same game repeatedly with feedback (e.g. Nyarko and Schotter [2002] and Rutstrom and Wilcox [2009]). In terms of analysis, we focus not just on rates of best response, but also on how these rates of best response vary across every neighborhood of stated beliefs. In addition to establishing that subjects’ beliefs are noisy, we show that within-subject variations in beliefs predict actions.

Studies on strategic sophistication, primarily using the level $k$ framework, typically use dominance solvable games to get around the non-identifiability issues discussed in Section 6.2.4 (e.g. Costa-Gomes and Crawford [2006] and Kneeland [2015]). Hence, little is known about how sophistication manifests itself in the important class of fully mixed games. By combining our rich subject-level beliefs data with a direct measure of sophistication, we provide some of the first evidence. In finding a correlation between sophistication measures from fully mixed and dominance solvable games, our analysis also suggests a “persistence of strategic sophistication” across these two domains, adding positive evidence to a literature that has found largely negative results (e.g. Georganas et al. [2015]).

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40This literature was jumpstarted with the negative results of Haile et al. [2008] who showed that structural QRE can rationalize the data from any one game without strong restrictions on the error distributions. Work since has focused on studying the restrictions imposed by other variants of QRE.

41Hyndman et al. [2013] provides some evidence in 3 × 3 games that beliefs toward the corners of the simplex are best responded to more often.
10 Conclusion

Motivated to contribute to a more realistic game theory, we study the beliefs people form over opponents’ behavior and the actions they take conditional on these beliefs.

We begin by characterizing an equilibrium model with “noise in actions” and “noise in beliefs”—a benchmark model that avoids unrealistic deterministic assumptions that would be trivially rejected. By injecting noise into both actions and beliefs, the model runs the risk of becoming vacuous, so we restrict both types of noise to satisfy axioms that are stochastic generalizations of “best response” and “correct beliefs”.

Using a laboratory experiment, we collect actions data and elicit beliefs for a canonical family of games with systematically varied payoffs. By having multiple elicitations per subject-game without feedback, our design allows us to (i) observe noise in both actions and beliefs and (ii) test the axioms of the benchmark model.

We find that both sources of noise are important for explaining features of the data, which suggests that deterministic assumptions may be an important source of misspecification. In particular, this calls into question the common practice of applying models with deterministic beliefs to experimental data.

Interestingly, despite the axioms being relatively weak, we find rejections. The most striking violation comes in the form of belief biases that depend on player role. Using a structural model applied to our subject-level beliefs data, we argue that the player role itself induces a higher degree of strategic sophistication in the player who faces more asymmetric payoffs and that this can explain the pattern of bias. This structural feature is not captured by any existing models and, in our view, merits further study.

References


Evan Friedman. Stochastic equilibria: Noise in actions or beliefs? *Working Paper*, 2019. 1, 3, 1, 2, 2.2, 10, 2.5, 2.5, 17, 9, 11.2


Jacob Goeree, Charles Holt, and Thomas Palfrey. Regular quantal response equilibrium. *Experimental Economics*, 2005. 1, 2, 2.1, 2.5, 2.5, 9, 11.2


Ben Greiner. Subject pool recruitment procedures: Organizing experiments with orsee. *Journal of the Economic Science Association*, 2015. 3.1


Edi Karni. A mechanism for eliciting probabilities. *Econometrica*, 2009. 3.3


Ariel Rubinstein and Martin Osborne. Sampling equilibrium, with an application to strategic voting. *Games and Economic Behavior*, 2003. 9


63
11 Appendix

11.1 Experimental instructions

Welcome!
This is an experiment in decision making, and you will be paid for your participation in cash. Different subjects may earn different amounts of money. What you earn depends partly on your decisions, partly on the decisions of others, and partly on luck. In addition to these earnings, each of you will receive $10 just for participating in and completing the experiment.

It is the policy of this lab that we are strictly forbidden from deceiving you, so you can trust the experiment will proceed exactly as we describe, including the procedures for payment.

The entire experiment will take place through your computers. It is important that you do not talk or in any way try to communicate with other subjects during the experiment.

Please turn off your cellphones now.

On the screen in front of you, you should see text asking you to wait for instructions, followed by a text box with a button that says “ID”. Your computer ID is the number at the top of your desk, which is between 1 and 24. In order to begin the experiment, you must enter your computer ID into the box and press ‘ID’. Please do that now.

You should all now see a screen that says “please wait for instructions before continuing”. Is there anyone that does not see this screen? This screen will appear at various points throughout the experiment. It is important that whenever you see this screen, you do not click ‘continue’ until told to do so.

The experiment has two sections. We will start with a brief instruction period for Section 1, in which you will be familiarized with the types of rounds you will encounter. Additional instructions will be given for Section 2 after Section 1 is complete.

If you have any questions during the instruction period, raise your hand and your question will be answered so everyone can hear. If any difficulties arise after the experiment has begun, raise your hand, and an experimenter will come and assist you.
At the beginning of the experiment, each subject will be assigned the color RED or the color BLUE. There will be an equal number of RED and BLUE subjects. If you are assigned RED, you will be RED for the entire experiment. If you are assigned BLUE, you will be BLUE for the entire experiment.

Section 1 consists of several rounds. I will now describe what occurs in each round. First, you will be randomly paired with a subject of the opposite color. Thus, if you are a RED subject, you will be paired with a RED subject. If you are a RED subject, you will be paired with a BLUE subject. You will not not know who you are paired with, nor will the other subject know who you are. Each pairing lasts only one round. At the start of the next round, you will be randomly re-paired.

[SLIDE 1]

In each round, you will see a matrix similar to the one currently shown on the overhead, though the numbers will change every round. In every round, you and the subject you are paired with will both see the same matrix, but remember that one of you is BLUE and one of you is RED.

Both subjects in the pair will simultaneously be asked to make a choice. BLUE will choose one of the two rows in the matrix, either ‘Up’ or ‘Down’, which we write as ‘U’ or ‘D’. RED will choose one of the two columns, either ‘Left’ or ‘Right’, which we write as ‘L’ or ‘R’. We refer to these choices as “actions”. Notice that each pair of actions corresponds to one of the 4 cells of the matrix. For instance, if BLUE chooses ‘U’ and RED chooses ‘L’, this corresponds to the top-left cell, and similarly for the others.

Thus, depending on both players’ actions, there are 4 possible outcomes:

- If BLUE chooses ‘U’ and RED chooses ‘L’, BLUE receives a payoff of 10, since that is the blue number in the UP–LEFT cell, and RED receives 20, since that is the RED number.
- If BLUE chooses ‘D’ and RED chooses ‘R’, BLUE receives a payoff of 11 and RED receives 75.
- And the other two cells UP–RIGHT and DOWN–LEFT are similar.

We reiterate: each number in the matrix is a payoff that might be received by one of the players, depending on both players’ actions. Are there any questions?

In this section, you will play for 20 rounds and 1 of your rounds will be chosen for your payment. This 1 round will be selected randomly for each subject, and the payment will depend on the actions taken in that round by you and the subject you were paired with. In the selected round, your payoff in the chosen cell denotes the probability with which you will receive $10. For example, if you receive a payoff of 60, then for that round you would receive $10 with 60% probability and $0 otherwise.

Since every round has an equal chance of being selected for payment, and you do not know which will be selected, it is in your best interest that you think carefully about all of your choices.

During the experiment, no feedback will be provided about the other player’s chosen action. Only at the end of the experiment will you get to see the round that was chosen for your payment and the actions taken by you and the player you were paired with in that round.

Before we begin the first section, you will answer 4 training questions to ensure you understand this payoff structure. In each of these 4 questions, you will be shown a matrix and told the actions chosen by both players. You will then be asked with what probability a particular player earns $10 if this round were to be selected for payment. That is, you are being asked for their payoff in the appropriate cell. To answer, simply type the probability as a whole number into the box provided and click ‘continue’. The page will only allow you to ‘continue’ when your answer is correct, at which point you may proceed to the next question. Please click ‘continue’ and answer the 4 training questions now.

[SLIDE 2]

Now that you’ve completed the training questions and understand the payoff matrices, we will proceed to Section 1. In each round of this section you will be randomly paired with another subject. If you are BLUE, you will be paired with a RED subject, and if you are RED, you will be paired with a BLUE subject. Recall that, at the start of each round, you will be randomly re-paired.

In each round, for each pair, the RED player’s task will be to select a column of the matrix, and the BLUE player’s task will be to select a row of the matrix, and these actions determine both players’ payoffs for the round.
[SLIDE 3]

You should now see an example round on the overhead. This shows the screen for a BLUE player, who is asked to choose between ‘U’ and ‘D’. Notice however that the text instructing you to make a choice is faded. This is because you must wait for 10 seconds before you are allowed to make a decision. Once 10 seconds has passed, the text will darken, indicating that you can now make a selection. The number of seconds remaining until you are able to choose is shown in the bottom right corner. Now the overhead shows what the screen will look like after the 10 seconds have passed.

[SLIDE 4]

The 10 seconds is a minimum time limit. There is no maximum time limit on your choices, and you should feel free to take as much time as you need, even after the 10 seconds has passed. In order to make your selection, simply click on the row or column of your choice. Once you have done so, your choice will be highlighted, and a ‘submit’ button will appear, as we now show on the overhead.

[SLIDE 5]

You may change your answer as many times as you like before submitting. If you would like to undo your choice, simply click again on the highlighted row or column. Once you are satisfied with your choice, click ‘submit’ to move on to the next round.

Before beginning the paid rounds of Section 1, we will play 4 practice rounds to familiarize you with the interface. These rounds will not be selected for payment. Are there any questions about the game, the rules, or the interface before we begin the practice rounds?

Please click ‘continue’ and begin the practice rounds now. You will notice that you have been assigned either RED or BLUE. This will be your color throughout the experiment. Please continue until you have completed the 4 practice rounds.

You have now completed the practice rounds, and we will proceed to the paid rounds of Section 1. Section 1 consists of 20 rounds, exactly like those you have just played. Recall that, in each round, you will be randomly paired with another subject and that one round will be randomly selected for payment. Are there any questions about the game, the rules, or the interface before we begin?

[SLIDE 6]

Please click ‘continue’ and play Section 1 now. The rules we discussed for Section 1 will be shown on the overhead as a reminder throughout.

[SLIDE 7]

We will now have a brief instruction period for Section 2, in which you will be familiarized with the types of rounds you will encounter.

If you have any questions during the instruction period, raise your hand and your question will be answered so everyone can hear. If any difficulties arise once play has begun, raise your hand, and an experimenter will come and assist you.

In this section, each round will be similar to those from Section 1. You will see some of the same matrices and your assignment of RED or BLUE will be the same as before.

Now, however, after being shown a matrix, your task will be to give your belief or best guess about the probability that a randomly selected subject chose a particular action when playing the same matrix in Section 1. That is, you will be shown a matrix, and the computer will randomly select a round from Section 1 in which the same matrix was played. Then,

- If you are RED, you will be asked for the probability that a randomly selected BLUE player chose ‘U’ in that round in Section 1.
- If you are BLUE, you will be asked for the probability that a randomly selected RED player chose ‘L’ in that round in Section 1.
As before, you will be paid for your responses. We will now describe this payment mechanism.

[SLIDE 8]

Consider first the matrix that is shown on the overhead. Please imagine that the computer has randomly selected a round from Section 1 in which this matrix was played. We wish to know your belief about the probability that a randomly selected RED player chose 'L' in that round. Please, take some time now to think carefully about what you believe this probability to be.

[SLIDE 9]

Consider the question that is now shown on the overhead, which asks which of the following you would prefer:

- Under Option A, you receive $5 if a randomly selected RED player chose 'L' in that round, and $0 otherwise.

- Under Option B, you receive $5 with probability 75%, and $0 otherwise.

Please think carefully about which of these two options you would prefer.

Presumably, if you believe the probability that a randomly selected RED player chose 'L' is greater than 75%, then you would prefer Option A, which you believe gives you the highest probability of a $5 prize. For example, if you believe this probability is 89%, you would choose Option A since 89 is greater than 75.

If, on the other hand, you believe the probability that a randomly selected RED player chose 'L' is less than 75%, then you would prefer Option B, which you believe gives you the highest probability of a $5 prize. For example, if you believe this probability is 22%, you would choose Option B since 22 is less than 75.

In this way, your answer to this question will tell us whether you believe this probability is greater than or less than 75%.

[SLIDE 10]

Now imagine we asked you 101 of these questions, with the probability in Option B ranging from 0% to 100%. Presumably you would answer each of these questions as described previously. That is, for questions for which the probability in Option B is below your belief, you would choose Option A, and for questions for which the probability in Option B is above your belief, you would choose Option B. Imagine, for example, you believe that there is a 64% probability that a randomly selected RED player chose 'L' in the selected round. Then, you would select Option A for all questions before #64, and Option B for all questions after #64. For Question #64, you could make either selection.

[SLIDE 11]

In this case, your selections would be as shown on the overhead, with the chosen options in black and the unchosen options in gray. From these answers, we could determine that you believe the probability that a randomly selected RED player chose 'L' is 64%.

In each round of this section, you will be faced with a table of 101 questions as shown on the overhead. To save time, instead of having you answer each question individually, we will simply ask you to type in your belief, and the answers to these 101 questions will be automatically filled out as above. That is, for rows of the table in which the probability in Option B is below your stated belief you will automatically select Option A, and for rows of the table in which the probability in Option B is at or above your stated belief you will automatically select Option B.

If this round is chosen for payment, one of the 101 rows of the table will be randomly selected and you will be paid according to your chosen option in that row. If you chose Option A in that row, a subject of the relevant color will be randomly chosen, and you will receive $5 if they played the relevant action in the selected round of Section 1. If you chose Option B in that row, you will receive $5 with the probability given in that option.

It is thus in your best interest, given your belief, to state your belief accurately. Otherwise, if you type something other than your belief, there will be rows of the table for which you will not be selecting the option that you believe gives you the highest probability of receiving a $5 prize.

In this section you will play 40 rounds, giving 40 such beliefs. At the end of the section, 2 rounds will be randomly chosen for payment. For each of these rounds, one of the 101 rows of the table will be randomly selected and you will be paid according to your chosen option in that row.
Are there any questions about this?

In addition to stating a belief, in each round you will also be asked to choose an action, as you did in Section 1. Now, however, the other action will not be determined by another subject acting simultaneously. Instead, recall that the computer has randomly selected a round from Section 1 featuring the matrix shown on your screen. The computer will also randomly select a player of the other color and record the action they took in that round. This is the action that you will be paired with. That is:

- If you are RED, the BLUE action will be that which a randomly selected BLUE player chose in the selected round of Section 1.
- If you are BLUE, the RED action will be that which a randomly selected RED player chose in the selected round of Section 1.

Again, the randomly selected round from Section 1 will feature the same matrix shown on your screen, so your payoff is determined as if you were paired with a randomly selected player from Section 1, rather than being paired with a player who chooses an action simultaneously.

As in Section 1, your payoff from taking an action gives the probability of earning $10 if the round is chosen for payment. At the end of the section, 2 rounds will be randomly chosen for payments based on your actions. This is in addition to the 2 rounds randomly chosen for payments based on your beliefs. Moreover, the randomization algorithm that selects these rounds will ensure that all 4 rounds feature different matrices and that these matrices will be different from that selected for payment in Section 1. In particular, this means that if a round is selected for an action-payment, it cannot also be selected for a belief-payment and vice versa.

As before, since you do not know which round will be selected for payment, nor which type of payment it will be selected for, these payment procedures ensure that, in each round, it is in your best interest to both state your belief accurately and choose the action that you think is best.

[SLIDE 12]

You should now see an example round on the overhead. This shows the screen for a BLUE player. As in Section 1, you will see the matrix in the middle of the screen. At the top of the screen, you are told that the computer has randomly selected a round of Section 1 in which this matrix was played.

Below this, the instructions are shown, and are again faded for 10 seconds. Once 10 seconds has passed, the text asking you for your belief will darken as now shown on the overhead.

[SLIDE 13]

You will not be able to select an action until after you have entered your belief.

Once you have entered your belief, the resulting probabilities will appear below or beside the matrix and the text asking you to select your action will darken, as now shown on the overhead.

[SLIDE 14]

Your belief must be a whole number between 0 and 100 inclusive. Once you enter your belief, we will automatically 'fill out' the questions in the 101 rows based on your belief as previously described. If you wish, at any time you may scroll down to observe the 101 rows.

As in Section 1, once you have selected an action, it will be highlighted on the matrix, as now shown on the overhead.

[SLIDE 15]

At this point, you may freely modify both your belief and action as many times as you wish before pressing 'submit'. Remember that there is no upper time limit on your choices, and you should feel free to take as much time as you need, even after the minimum 10 seconds has passed.

Before beginning the paid rounds of Section 2, we will play 3 practice rounds to familiarize you with the interface. These rounds feature the same matrices as the practice rounds from Section 1, and will not be selected for payment. Are there any questions about the game, the rules, or the interface before we begin the practice rounds?

Please click 'continue' to be taken to the first practice round now. Recall that your belief must be a whole number between 0 and 100 inclusive, and at any time you may scroll down to see the table of 101 questions. Please continue until you have completed the 3 practice rounds.

You've now completed the practice rounds, and we will proceed to the paid rounds of Section 2.
Recall that Section 2 consists of 40 rounds, exactly like those you have just played. 4 rounds will be randomly selected for payment–2 rounds for beliefs and 2 rounds for your actions. Again, these 4 rounds will feature different matrices to each other and to the matrix selected for payment in Section 1. The payment procedures ensure that it is always in your best interest to both state your belief accurately and choose the action that you think is best. Unlike Section 1, Section 2 will be played at your own pace without waiting for other subjects between rounds. Once you have completed Section 2, please remain seated quietly until all subjects have finished.

Are there any questions about the game, the rules, or the interface? If you have any questions during the remainder of the experiment, raise your hand, and an experimenter will come and assist you. You may click 'continue' and play Section 2 now. The rules we discussed for Section 2 will be shown on the overhead as a reminder throughout.

You have now completed the experiment. All that remains is to organize payments. To do this, you will be shown a page with all of your randomly selected rounds and your earnings in each. This page will also show you how to fill out the payment receipt at your desks. Before reaching this page, you will see an explanation page describing how the results are determined and how to read them. You may click 'continue' now and read through the explanation page. Then continue to the payments page, where you will see your results and fill out your receipt.

11.2 Proofs

Proof of Proposition 1. Existence follows from Brouwer’s fixed point theorem after showing \( \Psi_i : [0,1] \rightarrow [0,1] \) is continuous. To this end, let \( \mu_i(\cdot|\sigma_i)_{\sigma_i \in [0,1]} \) be the family of Borel measures derived from \( F_i(\cdot|\sigma_i)_{\sigma_i \in [0,1]} \) that define the belief distributions. From (B1) and (B2), \( \mu_i(\cdot|\sigma_i)_{\sigma_i \in [0,1]} \) has the property that \( \mu_i(B|\sigma_i) \) is continuous in \( \sigma_i \in (0,1) \) for any Borel subset \( B \in B([0,1]) \). From this and the fact that \( Q_i \circ \tilde{u}_i(\sigma_i) \) is continuous in \( \sigma_i \in [0,1] \), it is immediate that \( \Psi_i(\sigma_i) \) is continuous in \( \sigma_i \in (0,1) \). So we need only consider \( \sigma_i \rightarrow 0^+ \) (the case of \( \sigma_i \rightarrow 1^- \) is similar). From (B4) and (B1) and (B2), there are discontinuities at the endpoints: \( \mu_i(\{0\}|\sigma_i) = 0 \) for \( \sigma_i > 0 \) but \( \mu_i(\{0\}|0) = 1 \). However, from (B1) and (B2), \( \mu_i(B|\sigma_i) \) is continuous as \( \sigma_i \rightarrow 0^+ \) if \( B = [0,\epsilon) \) or \( B = (\epsilon, \epsilon_2) \) (i.e. if \( B \) or its complement is well-separated from 0), which implies that \( \mu_i([0,\epsilon]|\sigma_i) \rightarrow 1 \) as \( \sigma_i \rightarrow 0^+ \) for any \( \epsilon > 0 \). Hence, \( \Psi_i(\sigma_i) \) is continuous since \( \Psi_i(0) = Q_i \circ \tilde{u}_i(0) \) and as \( \sigma_i \rightarrow 0^+ \), beliefs become arbitrarily concentrated within a neighborhood of 0 and \( Q_i \circ \tilde{u}_i(\sigma_i) \) is continuous in \( \sigma_i \in [0,1] \).

Proof of Proposition 2. From (A3) and (B3), \( \Psi_U(\sigma_L) \) is strictly increasing in \( \sigma_L \) for any belief realizations \( \sigma_L'' > \sigma_L' \). \( Q_U(\tilde{u}_1(\sigma_L'')) > Q_U(\tilde{u}_1(\sigma_L')) \) by (A3), and if \( \sigma_L' \) increases, the distribution of \( \sigma_L' \) increases in the sense of stochastic dominance by (B3). Similarly, \( \Psi_U(\sigma_U) \) is strictly decreasing in \( \sigma_U \) and so the equilibrium is unique.

Proof of Proposition 3. Suppose \( \sigma_L < \sigma_L^{NE} \). By (B4), it must be that \( F_1(\sigma_L|\sigma_L) = \frac{1}{2} \), and hence \( \mathbb{P}(\sigma_L^U|\sigma_L) < \sigma_L^{NE} \) \( = \frac{1}{2} - \mathbb{P}(\sigma_L|\sigma_L) < \sigma_L^{NE} \) \( < 0, \frac{1}{2} \). By (A4), \( Q_U \circ \tilde{u}_1(\sigma_L) \in (0, \frac{1}{2}) \) for belief realization \( \sigma_L < \sigma_L^{NE} \) and \( Q_U \circ \tilde{u}_1(\sigma_L) \in (\frac{1}{2}, 1) \) for belief realization \( \sigma_L' > \sigma_L^{NE} \). Together, this implies that \( \Psi_U(\sigma_L) \in (0, \frac{1}{2}) \) for \( \sigma_L < \sigma_L^{NE} \). Using similar arguments, it must be that \( \Psi_U(\sigma_L) \) must satisfy (1) for all \( \sigma_L \). Conversely, let \( \sigma_L < \sigma_L^{NE} \) and \( c \in (0, \frac{1}{2}) \) be arbitrary. \( \Psi_U(\sigma_L) = c \) can be obtained by setting \( \sigma_L^U(\sigma_L) = \left\{\begin{array}{ll}0 & w.p. \frac{1}{2} \\1 & w.p. \frac{1}{2}\end{array}\right.\) and \( Q_U \) so that \( \frac{1}{2} Q_U \circ \tilde{u}_1(0) + \frac{1}{2} Q_U \circ \tilde{u}_1(1) = c \). The only restrictions are that \( Q_U \circ \tilde{u}_1(0) \in (0, \frac{1}{2}) \) and \( Q_U \circ \tilde{u}_1(1) \in (\frac{1}{2}, 1) \), so this is feasible. This construction violates (B1) and (B2), but can be modified to satisfy these axioms by smoothing out the distribution of \( \sigma_L^U(\sigma_L) \) arbitrarily little. As \( \sigma_L \) increases to \( \sigma_L' < \sigma_L^{NE} \), this construction can be extended so that \( \Psi_U(\sigma_L') = c' \) for any \( c' \in (c, \frac{1}{2}) \) in such a way that none of the axioms are violated. Proceeding in this fashion, any \( \Psi_U : [0,1] \rightarrow [0,1] \) that is continuous, strictly increasing, and satisfying (1) can be induced for some \{\( Q_U, \sigma_L'\)\}. Part (ii) is similar, and part (iii) follows since the QNBE is the intersection of the constructed \( \Psi_U \) and \( \Psi_L \).

Proof of Proposition 4. As \( X \) increases, \( Q_U(\tilde{u}_1(\sigma_L')) \) strictly increases for any \( \sigma_L' \). Hence, \( \Psi_U(\sigma_L) \) shifts up strictly. Since \( \Psi_U(\sigma_L) \) is strictly increasing and \( \Psi_L(\sigma_U) \) is strictly decreasing, it must be that \( \sigma_L^{QNE} \) strictly increases and \( \sigma_L^{QNE} \) strictly decreases (see Figure 1).
Proof of Proposition 5. The only if direction follows immediately from Propositions 3 and 4. We omit the if direction because it is very similar to that in the proof of Proposition 6 below as it basically combines the results for QRE and NBE.

Proof of Proposition 6. The only if direction can be found for very similar games for QRE in Goeree et al. [2005] and for NBE in Friedman [2019] in terms of \( X \). Since \( \sigma_{L_{1}}^{E} \) is strictly decreasing in \( X \), the result follows. The if direction is novel. For QRE, it is very simple. Let \( \{ \tilde{\sigma}_{L}^{X}, \tilde{\sigma}_{L}^{X} \}_{X} \) satisfy (i)-(iv). For player 1, set \( Q_{U} \) such that \( Q_{U} \circ \tilde{u}_{L_{1}}^{X} (\tilde{\sigma}_{L}^{X}) = \tilde{\sigma}_{L}^{X} \) for all \( X \) and similarly for \( Q_{L} \). It is easy to check that, because \( \{ \tilde{\sigma}_{L}^{X}, \tilde{\sigma}_{L}^{X} \}_{X} \) satisfies (i)-(iv), this does not violate (A3) or (A4). Since this only pins down \( Q \) at finite points, the rest of \( Q \) can be constructed in a way that satisfies (A1)-(A4). For NBE, it is more involved. Let \( \{ \tilde{\sigma}_{L}^{X}, \tilde{\sigma}_{L}^{X} \}_{X} \) satisfy (i)-(iv). For player 1, \( \Psi_{L_{1}}^{X} (\sigma_{L}) = 1 - F_{L_{1}}(\sigma_{L_{1}}^{X_{L_{1}}} | \sigma_{L}^{X}) \), so we must construct a family of CDFs \( F_{1}(\cdot) \) such that \( 1 - F_{1}(\sigma_{L_{1}}^{X_{L_{1}}} | \sigma_{L}^{X}) = \tilde{\sigma}_{L}^{X} \) for all \( X \) and that satisfies (B1)-(B4).

We illustrate this construction in Figure 20. Given that \( \{ \tilde{\sigma}_{U}^{X}, \tilde{\sigma}_{L}^{X} \}_{X} \) satisfies (i)-(iv), we have that (1) \( \tilde{\sigma}_{L}^{X} < \tilde{\sigma}_{L}^{X_{L_{1}}^{X}} \) whenever \( X > X_{L_{1}}^{X} \), (2) \( \frac{1}{2} > \tilde{\sigma}_{L}^{X} > \sigma_{L_{1}}^{X_{L_{1}}} \) if \( X > 20 \) and \( \frac{1}{2} < \tilde{\sigma}_{L}^{X} < \sigma_{L_{1}}^{X_{L_{1}}} \) if \( X < 20 \), and (3) \( \tilde{\sigma}_{L}^{X} > \frac{3}{4} \) if \( X > 20 \) and \( \tilde{\sigma}_{L}^{X} < \frac{1}{2} \) if \( X < 20 \). As illustrated in the figure, this allows us to construct, for each \( \tilde{\sigma}_{L}^{X} \), a CDF \( F_{1}(\cdot) \) that satisfies \( 1 - F_{1}(\sigma_{L_{1}}^{X_{L_{1}}} | \tilde{\sigma}_{L}^{X}) = \tilde{\sigma}_{L}^{X} \), meaning it can generate the data, and also such that each \( F_{1}(X| \tilde{\sigma}_{L}^{X}) \) is strictly increasing in \( z \in \{0, 1\} \), \( F_{1}(z| \tilde{\sigma}_{L}^{X}) < F_{1}(z| \tilde{\sigma}_{L}^{X}) \) for all \( z \in \{0, 1\} \) if \( \tilde{\sigma}_{L}^{X} < \tilde{\sigma}_{L}^{X} \), and \( F_{1}(\tilde{\sigma}_{L}^{X} | \tilde{\sigma}_{L}^{X}) = \frac{1}{2} \) for all \( X \). Hence, the constructed \( \{ F_{1}(\cdot) \}_{X} \) satisfies (B1)-(B4) and can be extended to \( \{ F_{1}(\cdot) \}_{\sigma_{L}^{X} \in \{0, 1\}} \). For the extension, order the values of \( X \) in the dataset: \( X_{1} > X_{2} > \ldots > X_{n} \) and \( \sigma_{L}^{X_{1}} < \sigma_{L}^{X_{2}} < \ldots < \sigma_{L}^{X_{n}} \). For \( \sigma_{L}^{X} \in \{ \sigma_{L}^{X_{1}}, \sigma_{L}^{X_{1}+1} \} \) set \( F_{1}(z| \tilde{\sigma}_{L}^{X}) = \alpha(X) F_{1}(z| \tilde{\sigma}_{L}^{X}) + (1 - \alpha(X)) F_{1}(z| \tilde{\sigma}_{L}^{X+1}) \) where \( \alpha(X) \) is such that \( F_{1}(\sigma_{L}^{X} | \tilde{\sigma}_{L}^{X}) = \frac{1}{2} \), which is uniquely defined. Finally, extending for \( \sigma_{L}^{X} < \tilde{\sigma}_{L}^{X_{1}} \) and \( \tilde{\sigma}_{L}^{X} > \sigma_{L}^{X_{n}} \) is straightforward.

Proof of Proposition 7. (i): Let \( w \) and \( v \) with \( w = f(v) \) for some concave \( f \). Let \( X > 20 \). Without loss, normalize so that \( w(0) = v(0) = 0 \) and \( w(X) = v(X) = 1 \). For arbitrary utility function \( u \), it is easy to show that \( \hat{\sigma}_{L_{1}}^{u_{X}} \left( \frac{w(20)}{w(20) + 1} \right) \). Since \( w \) is more concave than \( v \), \( w(20) > v(20) \) and thus \( \hat{\sigma}_{L_{1}}^{u_{X}} > \hat{\sigma}_{L_{1}}^{v_{X}} \). Similarly, if \( X < 20 \), normalize without loss so that \( w(0) = v(0) = 0 \) and \( w(20) = v(20) = 1 \). This implies that \( \hat{\sigma}_{L_{1}}^{v_{X}} < \frac{1}{1 + w(X)} \). Since \( w \) is more concave than \( v \), \( w(20) > v(20) \) and thus \( \hat{\sigma}_{L_{1}}^{v_{X}} < \hat{\sigma}_{L_{1}}^{u_{X}} \). Part (ii) is the same, except with \( v(z) = z \), which implies \( \hat{\sigma}_{L_{1}}^{v_{X}} = \frac{20}{20 + X} = \sigma_{L_{1}}^{E_{X}} \).
Figure 20: Construction of belief-map in proof of Proposition 6. This illustrates the constructed CDFs of player 1’s beliefs to rationalize as NBE the actions dataset \( \{ \hat{X}_U, \hat{X}_L \} \) for \( X \in \{ X', X'' \} \) with \( X' > 20 \) and \( X'' < 20 \). The purple CDF is \( F_1(\cdot|\hat{X}_L) \) and the green CDF is \( F_1(\cdot|\hat{X}_L) \).
11.3 Details of statistical tests

For what follows, let $N^S_i$ denote the set of subjects in role $i \in \{1,2\}$ in session $S \in \{[A,BA],[A,A]\}$.

Bowman et al. (1998) test. The null hypothesis is that a regression function, which in our case is the expected action conditional on beliefs, is weakly monotone.

Let $b^{ix}_{il}$ be the $l$th of 5 belief statements for subject $s$ in role $i$ for game $x \in G$ where $G$ is a set of games that is either the entire set of six games $\{X80,X40,\ldots,X1\}$ or any one of these games. Let $a^{ix}_{il} \in \{0,1\}$ be the action corresponding to belief $b^{ix}_{il}$. The method is based on running non-parametric regressions of $a^{ix}_{il}$ on $b^{ix}_{il}$, and we denote such estimators by $\hat{m}(b)$ and $\hat{w}(b)$. We describe the process when aggregating across all subjects of type $i$ indexed by set $I$, which can be either all subjects $N^1_i[A,BA]$ or a single subject $\{s\}$.

- Step 1: Find the critical local linear (lowess) regression bandwidth $h_c$, which is the smallest such that $\hat{m}(b; h_c)$ is weakly monotone (increasing if $i = 1$, decreasing if $i = 2$).
- Step 2: For each $s, x, k$, calculate $\hat{e}^{ix}_{il} = a^{ix}_{il} - \hat{w}(b^{ix}_{il}; h_0)$ where $\hat{w}$ is some estimator with bandwidth $h_0$ chosen to minimize the mean integrated squared error $\sum_{s,x,i}(\hat{m}(b^{ix}_{il}) - m(b^{ix}_{il}))$ where $a^{ix}_{il} = m(b^{ix}_{il}) + \hat{e}^{ix}_{il}$ is the true model.\(^{42}\)
- Step 3: Resample the subjects in $I$ with replacement $|I|$ times. Conditional on drawing each subject $s$, resample 5 times with replacement from $\{\hat{e}^{ix}_{il}, \hat{e}^{ix}_{il}, \ldots, \hat{e}^{ix}_{il}\}$ for each $x \in |G|$ for a total bootstrap sample of $5 \cdot |G| \cdot |I|$ observations $\{\hat{e}^{ix}_{il}\}_{i \in \{1,2,\ldots,30\} \cdot |I|}$ and thus $\{\hat{a}^{ix}_{il} = \hat{m}(b^{ix}_{il}; h_c) + \hat{e}^{ix}_{il}\}_{i \in \{1,2,\ldots,30\} \cdot |I|}$ where $b^{ix}_i$ is the belief associated with $\hat{e}^{ix}_i$.
- Step 4: Apply $\hat{m}$ using $h_c$ to $\{\{\hat{a}^{ix}_i\}_{i \in \{1,2,\ldots,30\} \cdot |I|}\}$ and observe whether or not the result is monotone.
- Step 5: Repeat Steps 3 and 4 $B = 5,000$ times, constructing the $p$-value by determining the proportion of estimates at Step 4 which are not monotonic (not anywhere increasing if $i = 1$, not everywhere decreasing if $i = 2$).

Abadie (2002) test. The null hypothesis is that $F_i(z|\sigma^x_j) \leq F_i(z|\sigma^y_j)$ for $z \in [0,1]$.

- Step 1: Compute Kolmogorov-Smirnov statistic $\hat{T} = \left(\frac{\|\hat{N}^{[A,BA]}_i\|}{\|\hat{N}^{[A,BA]}_i\| + \|\hat{N}^{[A,A]}_i\|}\right)^{\frac{1}{2}} \cdot \sup_{t \in [0,1]} \left(\hat{F}_i(z|\sigma^x_j) - \hat{F}_i(z|\sigma^y_j)\right)$, where $\hat{F}_i(z|\sigma^x_j)$ and $\hat{F}_i(z|\sigma^y_j)$ are the empirical CDFs of beliefs.
- Step 2: Resample the subjects in $N^{[A,BA]}_i$ with replacement. Conditional on drawing each subject, resample with replacement from her 10 belief statements, i.e. pooled together from $x$ and $y$, and assign the first 5 to group $x$ and the second 5 to group $y$. Do this $\|\hat{N}^{[A,BA]}_i\|$ times to form two bootstrapped CDFs $\hat{F}_i(z|\sigma^x_j)_b$ and $\hat{F}_i(z|\sigma^y_j)_b$ where $b$ is the bootstrap index. Use these to calculate $T^*_b$.
- Step 3: Repeat previous step $B = 5,000$ times.
- Step 4: Calculate $p$-value as $\sum_{b=1}^B 1\{T^*_b > \hat{T}\}/B$.

\(^{42}\)Bowman et al. [1998] suggest to use lowess regression with bandwidth selected by the method of Ruppert et al. [1995]. We opt instead for local linear kernel regression with cross-validation based bandwidth selection for its wider availability in statistical packages (Stata 15.0 command `npregress kernel`, which uses optimal bandwidth selection by default).
11.4 The effects of belief elicitation: a closer look

In the top panels of Appendix Figure 21, we plot the action frequencies from \([A,BA]\) and \([A,BA]\). That is, we are comparing first-stage actions (without belief elicitation) to second-stage actions, each of which was preceded by belief elicitation.\(^{43}\) For both players, we observe systematic and significant differences between the two stages. This is confirmed in the first 2 columns of Appendix Table 13 in which we regress actions on indicators for each of the six \(X\)-games (omitted from the table) and indicators for each of the six games interacted with an indicator for the \(f\) second-stage. \(F\)-tests reject that the action frequencies are the same across stages.

Our hypothesis is that these differences are caused by belief elicitation. However, the two stages differ in which came first, the fact that the games in the second stage are played against previously recorded actions, the number of rounds, and very slightly in their composition of games. To pin down the cause, we ran the additional \([A,A]\) treatment. This is identical to the \([A,BA]\) treatment except beliefs are not elicited (and instructions never mention belief elicitation).

The bottom panels of Appendix Figure 21 plot the action frequencies from \([A,A]\) and \([A,A]\), and Columns 3-4 of Appendix Table 13 replicate columns 1-2 for the \([A,A]\) treatment. We find that the actions data is statistically indistinguishable between the two stages of the \([A,A]\) treatment. In particular, the difference between player 1’s first- and second-stage action frequencies completely disappears. We conclude that belief elicitation does have an effect on actions.

Our goal in this paper is to study the relationships between beliefs and the associated actions without our own interference as experimenters. How does the fact that belief elicitation affects second-stage actions influence our conclusions? This depends on what is driving the effect.

There are two channels through which belief elicitation may have an effect on the actions subjects take. It could be that (i) elicitation affects beliefs or (ii) elicitation affects actions conditional on beliefs. If only the former “beliefs channel” is active, only testing axioms on the belief-map would be affected since we condition on second-stage beliefs when testing axioms on the action-map. If only the latter “actions channel” is active, only testing axioms on the action-map would be affected since we are comparing second-stage beliefs to first-stage actions when testing axioms on the belief-map. Since we do not observe the beliefs subjects had in the first stage, there is no way of knowing definitively the degree to which either channel is active, but our previous analysis gives insight.

From the top panels of Appendix Figure 21, the direction of the change in actions due to belief elicitation is systematic. For player 1, subjects are more likely to choose \(D\) for \(X > 20\) and \(U\) for \(X < 20\). For player 2, subjects are more likely to choose \(R\) for \(X > 20\) and \(L\) for \(X < 20\). While the effect is systematic for both players, the effect for player 2 is rather small.

Suppose the elicitation effect is through the beliefs channel. In the context of our structural model, this is consistent with increased sophistication for player 1, with \(\hat{\sigma}^{[A,BA]}_{U}\) resembling the first-stage actions of the high sophistication group in Figure 14. For player 2, the effect is consistent with decreased sophistication. We find this plausible for player 1 only as it is intuitive that eliciting beliefs may induce subjects to think more deeply about opponents’ behavior.

Suppose the elicitation effect is through the actions channel. Player 2’s actions become more extreme, so it could be that belief elicitation simply reduces the probability that player 2-subjects make mistakes or “trembles” for any given belief, which we find very plausible. For player 1, this would have an effect in the opposite direction from that which we observe, so it may be at play, but is overwhelmed by an effect in the opposite direction.

For player 1, we believe the beliefs channel dominates. If this is the case, prior to elicitation, player 1-subjects had beliefs closer to the uniform distribution and thus beliefs with a more conservative bias. For player 2, we believe the actions channel dominates, so prior to elicitation, player 2-subjects’ actions would have been noisier for any given belief. Under these interpretations, all of our main conclusions would be unchanged.

If belief elicitation affects actions largely through beliefs by making subjects more sophisticated, a natural hypothesis is that more sophisticated subjects will be less affected by belief elicitation. Since we have a measure of sophistication from the dominance solvable games, this is easy to test. In Appendix Figure 22, we plot the first- and second-stage action frequencies from \([A,BA]\) by the sophistication groupings from Section 6.2.3. This seems to indicate that the effect of elicitation is primarily driven by subjects with low sophistication, and this is confirmed in Appendix Table 14.

\(^{43}\)The results are similar if, instead, we compare the data from \([A,o]\) and \([A,BA]\), but this would be somewhat confounded by composition effects.
Table 13: Effects of belief elicitation. We regress actions on indicators for all six X-games (omitted) and indicators for each of the six games interacted with an indicator for the second stage. Columns 1-2 are for [A,BA], and columns 3-4 are for [A,A]. We also report the results of $F$-tests of the hypothesis that all six coefficients are zero. Standard errors are clustered at the subject-game level.

<table>
<thead>
<tr>
<th></th>
<th>[A,BA]</th>
<th>[A,A]</th>
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<tr>
<td></td>
<td>(1) $\hat{\alpha}_U$</td>
<td>(2) $\hat{\alpha}_L$</td>
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<tr>
<td>2nd stage $\times$ X80</td>
<td>-0.119**</td>
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<tr>
<td></td>
<td>(0.030)</td>
<td>(0.156)</td>
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<td>2nd stage $\times$ X40</td>
<td>-0.019</td>
<td>-0.059</td>
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<tr>
<td></td>
<td>(0.748)</td>
<td>(0.111)</td>
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<tr>
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<td>0.130**</td>
<td>0.105**</td>
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<tr>
<td></td>
<td>(0.013)</td>
<td>(0.031)</td>
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<td>0.007</td>
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<tr>
<td></td>
<td>(0.000)</td>
<td>(0.850)</td>
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<tr>
<td>2nd stage $\times$ X2</td>
<td>0.070</td>
<td>0.091**</td>
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<tr>
<td></td>
<td>(0.202)</td>
<td>(0.040)</td>
</tr>
<tr>
<td>2nd stage $\times$ X1</td>
<td>0.124**</td>
<td>0.102**</td>
</tr>
<tr>
<td></td>
<td>(0.037)</td>
<td>(0.016)</td>
</tr>
</tbody>
</table>

$F$-test

|                  | 5.12***  | 3.24***  | 0.21     | 0.92   |
|                  | (0.000)  | (0.004)  | (0.972)  | (0.485) |

$d1,d2$ [6,323] [6,335] [6,161] [6,161]

Observations 2592 2676 1134 1134

$p$-values in parentheses

* $p < .1$, ** $p < .05$, *** $p < .01$
Figure 21: Effects of belief elicitation. The top panels plot first-stage and second-stage actions from [A,BA], and shows a systematic difference between the two stages. The bottom panels plot first-stage and second-stage actions from [A,A], and shows no difference between the stages.
Figure 22: Effects of belief elicitation by sophistication group.
Table 14: Effects of belief elicitation by sophistication group. For high and low sophistication groups and for each player, we regress actions from both stages of [A,BA] on indicators for all six X-games (omitted) and indicators for each of the six games interacted with an indicator for the second stage. Columns 1-2 are for the high sophistication group, and columns 3-4 are for the low sophistication group. We also report the results of $F$-tests of the hypothesis that all six coefficients are zero. Standard errors are clustered at the subject-game level.
11.5 Experimental interface

Figure 23: Screenshots from first stage. This figure shows an example round from the perspective of a player 1-subject (blue). At the start of the round, the subject sees the payoff matrix (left screen), and a 10 second timer counting down to 0 (not shown here) is seen at the bottom right corner of the screen. After 10 seconds pass, the text “Please click to select between U and D:” darkens (middle screen) indicating that the subject may take an action. To select an action, the subject clicks on a row of the matrix. The row becomes highlighted and a ‘Submit’ button appears (right screen). At this point, the subject may freely modify his answer before submitting. The subject may undo his action choice by clicking again on the highlighted row.
**Figure 24:** Screenshots from second stage of $\{A,BA\}$. This figure shows an example round from the perspective of a player 1-subject (blue). At the start of the round, the subject sees the payoff matrix (top-left screen) and is told “The computer has randomly selected a round of Section 1 in which the below matrix was played.” After 10 seconds pass, the text “What do you believe is the probability that a randomly selected red player chose L in that round?” darkens (top-right screen) indicating that the subject may state a belief. The subject enters a belief as a whole number between 0 and 100. Once the belief is entered, the corresponding probabilities appear below the matrix and the text “The computer has randomly selected a red player and recorded their action from that round. Please click to select between U and D:” darkens (bottom-left screen) indicating that the subject may take an action. Only after stating a belief may the subject select an action, but after the belief is stated, the subject may freely modify both his belief and action before submitting. After a belief is entered and an action is selected, the ‘Submit’ button appears (bottom-right screen).
Figure 25: Screenshots from second stage of [A,A] The first stage [A,A] is identical to that of the [A,BA]. The second stage of the [A,A] is the same as that of [A,BA], except beliefs are not elicited.
11.6 Additional Figures

Figure 26: QNBE and the data. The green dot gives the empirical action frequencies from $[A, o]$, the red square gives the median belief, and the black diamond is the Nash equilibrium.
Figure 27: Subjects’ rates of best response. This figure gives histograms of subjects’ rates of best response across all X-games. The solid lines are averages, and the dashed lines in the bottom panel mark the average rate of best response from Nyarko and Schotter [2002].
Figure 28: Action frequencies predicted by beliefs. We plot the predicted values (with 90% error bands) from restricted cubic spline regressions of actions on beliefs (4 knots at belief quintiles, std. errors clustered by subject) superimposed over belief histograms. The vertical dashed line is the indifferent belief, and the horizontal line is set to one-half.

\[
\hat{Q}_U(\bar{u}_1(\sigma'_L)) \quad \hat{Q}_L(\bar{u}_2(\sigma'_U))
\]
Figure 29: *Concave utility explains monotonicity failures.* For player 1 and each of the $X$-games, we plot the predicted values (with 90% error bands) from restricted cubic spline regressions of actions on beliefs (4 knots at belief quintiles, standard errors clustered by subject). Belief histograms appear in gray, the vertical dashed line is the risk neutral indifferent belief $\sigma^*_j = \sigma^*_{j,NE}$, and the horizontal line is set to one-half. The solid vertical line is the indifferent belief with concave utility that is estimated from fitting a single curvature parameter to all player 1-subjects’ data.
Figure 30: Average time to form beliefs by game and player. We plot the average time until stated beliefs are finalized by game and player role.
Figure 31: Belief distributions by sophistication group. The left panel is for player 2’s beliefs about $U$, and the right panel is for player 1’s beliefs about $L$. The colored histogram is for the high sophistication group, and the white histogram is for the low sophistication group. Mean beliefs are given as colored and black lines, for high and low sophistication, respectively.
### 11.7 Additional Tables

Table 15: Rates of best response. This table reports the average rates of best response by player and game. Significance is based on a two-sided $t$-test of the null hypothesis that the rate of best response is one-half. Standard errors are clustered by subject.

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<td>0.667***</td>
<td>0.600**</td>
<td>0.544</td>
<td>0.544</td>
<td>0.639***</td>
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$p$-values in parentheses
* $p < .1$, ** $p < .05$, *** $p < .01$

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$p$-values in parentheses
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<th>L</th>
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</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>40</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 16: Additional games.
<table>
<thead>
<tr>
<th></th>
<th>(1) quintile</th>
<th>(2) equally spaced</th>
</tr>
</thead>
<tbody>
<tr>
<td>very low beliefs</td>
<td>-0.006***</td>
<td>-0.006***</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>low beliefs</td>
<td>-0.005***</td>
<td>-0.004***</td>
</tr>
<tr>
<td></td>
<td>(0.009)</td>
<td>(0.010)</td>
</tr>
<tr>
<td>medium beliefs</td>
<td>-0.006***</td>
<td>-0.010***</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>high beliefs</td>
<td>-0.009***</td>
<td>-0.010***</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.000)</td>
</tr>
<tr>
<td>very high beliefs</td>
<td>-0.005***</td>
<td>-0.005***</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.003)</td>
</tr>
</tbody>
</table>

Observations 1680 1680

*p-values in parentheses
* $p < .1$, ** $p < .05$, *** $p < .01$

Table 17: Fixed effect regressions of actions on beliefs–player 2, pooled across games. For player 2, we pool together the data from all six games. In column 1, we divide the individual belief statements into quintiles—very low, low, medium, high, and very high beliefs. For each belief quintile, we run a separate linear regression of actions on beliefs that are both first demeaned by subtracting subject-specific averages. In column 2, we do the same thing, except the five belief groups are based on evenly spaced bins of 20 belief points. Standard errors are clustered by subject.
Table 18: Bias in beliefs. This table reports, for each player and game, the empirical bias in beliefs as measured by the difference between the median or mean belief statement and the empirical action frequency (expressed as percentages). In both cases, we report the $p$-values from two-sided tests of the null hypothesis that beliefs are unbiased. When using the median, $p$-values are bootstrapped in a way so as to preserve the within-subject correlation observed in the data. When using the mean, we use standard $t$-tests, clustering by subject.
Table 19: The sophistication gap. In column 1, we regress $\hat{\beta}(k \geq 2)$ on an indicator for player 1. Column 2 controls for subject-average response time on the three rounds of $Di$ (since sophistication is measured entirely with beliefs data, we use the time until stated beliefs are finalized). Column 3 additionally controls for $\hat{\beta}(k \geq 1)$. Columns 4-6 are the same, except we first drop subjects who ever took a dominated action in $Di$ throughout the experiment.
### Table 20: Sophistication and behavior

We regress beliefs or actions on indicators for all six $X$-games (omitted) and indicators for each of the six games interacted with an indicator for the high sophistication group. Columns 1-2 use player 1-subjects and columns 3-4 use player 2-subjects. We also report the results of $F$-tests of the hypothesis that all six coefficients are zero. Standard errors are clustered at the subject-game level.

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(\hat{\sigma}_L)$</td>
<td>$(\hat{\sigma}_U)$</td>
</tr>
<tr>
<td>High soph. $\times X_{80}$</td>
<td>-20.172*** (0.000)</td>
<td>-0.297*** (0.006)</td>
</tr>
<tr>
<td>High soph. $\times X_{40}$</td>
<td>-14.359*** (0.001)</td>
<td>-0.304*** (0.001)</td>
</tr>
<tr>
<td>High soph. $\times X_{10}$</td>
<td>10.107** (0.032)</td>
<td>0.335*** (0.001)</td>
</tr>
<tr>
<td>High soph. $\times X_{5}$</td>
<td>15.687*** (0.001)</td>
<td>0.309*** (0.005)</td>
</tr>
<tr>
<td>High soph. $\times X_{2}$</td>
<td>18.808*** (0.000)</td>
<td>0.330*** (0.003)</td>
</tr>
<tr>
<td>High soph. $\times X_{1}$</td>
<td>17.975*** (0.001)</td>
<td>0.231** (0.025)</td>
</tr>
</tbody>
</table>

$F$-test: 11.91*** (0.000) 8.53*** (0.000) 5.85*** (0.000) 0.51 (0.801)

Observations: 3240 648 3360 672

$p$-values in parentheses

* $p < .1$, ** $p < .05$, *** $p < .01$